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SINGULAR CAUCHY INTEGRALS AND CONFORMAL WELDING ON JORDAN CURVES

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Abstract. Let γ_t denote a one-parameter family of Jordan curves on the Riemann sphere depending real-analytically on the parameter t. Let D_t and $D_t^\star$ be the two complementary Jordan regions separated by γ_t, and F_t and G_t denote, respectively, the (normalized) Riemann mappings of the unit disc onto these regions. The conformal welding homeomorphism associated to γ_t, $\omega_t = {F_t}^{-1} \circ G_t$, is a self-homeomorphism of S^1 obtained by comparing the boundary values of the two complementary Riemann mappings; it plays an important role in geometric function theory and in the theory of Teichmüller spaces. In this paper we show how to determine the t-expansions for F_t and G_t, up to arbitrary order in t, from knowledge of the ω_t alone.

It appears interesting that Cauchy singular integrals and the Plemelj–Sokhotski "double-layer" formulae play the crucial role in solving this problem. In an earlier paper in this journal [N2], we had derived a formula for the first derivative of the conformal welding correspondence from the space of quasicircles to the space of quasisymmetric homeomorphisms. Our present work reproves that result and goes on to provide formulas for the higher derivatives.

1. Introduction

Given any oriented Jordan curve γ in the complex plane, there are associated to it the following three fundamental objects:
(1) Riemann mapping $F\colon \Delta \to \operatorname{int}(\gamma)$;
(2) Riemann mapping $G\colon \Delta^\star \to \operatorname{ext}(\gamma)$;
(3) Conformal welding homeomorphism $\omega\colon S^1 \to S^1$ comparing the boundary homeomorphisms of F and G, i.e., $\omega = F^{-1} \circ G$ on S^1.

Notations. Δ denotes the open unit disc, $\Delta^\star$ its exterior in the Riemann sphere $\mathbf{P}^1$, and S^1 is the unit circle, $S^1 = \partial\Delta = \partial D^\star$; $\operatorname{int}(\gamma) = D$ and $\operatorname{ext}(\gamma) = D^\star$ are the two complementary Jordan regions determined by γ on the Riemann sphere. We assume $\infty \in D^\star$.

By Caratheodory's theorem one knows that F and G extend continuously to S^1 providing two natural parametrizations of the curve γ—and the welding homeomorphism compares them. (In this discussion we are ignoring the innocuous Möbius transformation ambiguity in the choice of the functions F, G and ω. That is easily taken care of by fixing normalizations.)

Theoretically speaking, knowledge of F or G determines, of course, the bounding curve γ itself, so that, assuming for example F alone to be known,

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the complementary mapping G and the welding ω should be determinable. *But there are no formulae known to effect these passages.*

Further, for quasicircles γ it is well-known that the welding ω is a quasisymmetric homeomorphism of the circle, and that is a complete invariant for the curve—in fact γ can be recovered from ω by a well-known "μ-trick". See, for instance, [A]. Therefore, at least in the context of the universal Teichmüller space, $\mathscr{T}(\Delta)$, comprising (Möbius equivalence classes of) quasicircles, (and possibly for more general classes of curves), *any one* of the three pieces of information above should, in principle, be sufficient to determine the other two. [It is, of course, quite trivial to get a formula for the third from knowledge of any two of the functions above.]

It may be worth pointing out here that the question of passing from information of a quasisymmetric welding homeomorphism γ to the associated Riemann mappings F or G, and vice versa, is quite fundamental in Teichmüller theory. Given a reference Riemann surface X, uniformized by a Fuchsian group Γ operating on Δ, any other (q.c. related) complex structure on it is encoded by some new uniformizing Fuchsian group Λ. Namely, $\tilde{X} = \Delta/\Lambda$ is another point of the Teichmüller space $\mathscr{T}(X)$ (which is a complex submanifold of $\mathscr{T}(\Delta)$), and, Λ turns out to be a quasisymmetric conjugate of the original group; i.e., $\Lambda = \omega\Gamma\omega^{-1}$, for some quasisymmetric homeomorphism ω of S^1. If one now understands which quasicircle γ produces this ω as its welding homeomorphism, and finds the Riemann mappings F (and/or G) to the interior and exterior of γ, then one can immediately determine the complex analytic variation of the moduli from $X = \Delta/\Gamma$ to $\tilde{X} = \Delta/\Lambda$. Indeed, the Kleinian group given by $K = G \circ \Gamma \circ G^{-1}$ will operate on the region interior to γ and produce the new surface $\tilde{X}$ as the quotient. See, for instance, [A], [N1]. Thus, passing from knowledge of ω to the Riemann map G entails basic understanding of the complex analytic Bers embeddings of the Teichmüller spaces.

In this paper we consider one parameter families of Jordan curves, γ_t, with corresponding F_t, G_t and ω_t. Let us assume that all the information for the initial (reference) curve γ_0 is known. We show that certain singular integrals on γ_0 with Cauchy kernel can be used to determine *all* the successive (first and higher order) variations of both F_t and G_t starting from knowledge of ω_t alone. It appears rather surprising and interesting that Cauchy singular integrals and the Plemelj–Sokhotski "double-layer" formulae play such a crucial role in solving this problem.

In an earlier paper [N2] we had derived a formula for the derivative of the conformal welding correspondence from the space of quasicircles to the space of quasisymmetric homeomorphisms. Our present calculations naturally allow us to reprove that result and go deeper.

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2. Singular Cauchy integrals on Jordan curves

Let γ be a rectifiable Jordan curve in the plane, and suppose that f is any continuous complex valued function defined on γ. As in the Introduction, let D and $D^\star$ denote the complementary simply-connected regions separated by γ on the Riemann sphere.

Question. Can we decompose f additively into two parts, $f = f^+ - f^-$, such that f^+ is the boundary values of a holomorphic function, say H^+, on D, and f^- is similarly the trace of a holomorphic function H^- on $D^\star$. (H^- is required to be analytic at ∞ also, of course.) The classical Plemelj–Sokhotski double-layer formulae provide an affirmative answer to this query whenever f is Hölder continuous. We will explain this below.

Remark. It is rather evident that a decomposition as above, if exists, must be *unique*—up to the choice of an additive constant. That is clear by assuming two such decompositions for f to exist and comparing the corresponding H^+ and H^- functions. Thus, normalizing $H^-(\infty) = 0$, absolutely fixes things.

Consider the holomorphic function on the union of D and $D^\star$ given by:

$$H(z) = (2i\pi)^{-1} \int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta - z} \tag{2.1}$$

Theorem 2.1. *Denote the restrictions of H to D and $D^\star$ by H^+ and H^-, respectively. Then H^+ and H^- both have non-tangential boundary values on γ, say f^+ and f^-. At any point σ on the curve γ one has:*

$$f^+(\sigma) = \tfrac{1}{2} f(\sigma) + (2i\pi)^{-1} \int_\gamma^{(CPV)} \frac{f(\zeta)\,d\zeta}{\zeta - \sigma} \tag{2.2+}$$

$$f^-(\sigma) = -\tfrac{1}{2} f(\sigma) + (2i\pi)^{-1} \int_\gamma^{(CPV)} \frac{f(\zeta)d\zeta}{\zeta - \sigma} \tag{2.2−}$$

Therefore, as desired:

$$f = f^+ - f^-. \tag{2.3}$$

Notation. The CPV superscript for the integrals in the formulae (2.2) indicates that these are *singular integrals* on γ, and the Cauchy principal value is being taken. That is done as follows: take a little disc of radius r centered at σ, and compute the integral indicated on the portion of γ that is *outside* this little disc. Then proceed to the limit as r tends to 0.

A proof of the above result can be seen in [G]. See also [Du], [D] for related material.

To understand the above theorem a few more remarks may be in order. Fix any p bigger than 1, and identify the usual L^p space of an interval with $L^p(\gamma)$— the L^p space of functions living on γ; (simply use the arc-length parametrization for γ). Then it turns out that whenever f is an element of the $L^p(\gamma)$ satisfying:

$$\int_\gamma z^k f(z)\,dz = 0, \qquad \text{for all } k \in \mathbf{N} \tag{2.4}$$

then f^- and H^- are identically zero, [and H^+ is actually a member of the Hardy space $H^p(D)$]. When γ is the unit circle S^1 itself, then of course this corresponds to the requirement that all the negative Fourier coefficients of f should vanish. In fact, in that case f^+, for arbitrary f, is nothing other than the sum of the Fourier expansion of f over the non-negative indices, and f^- is the (negative of) the Fourier sum over the negative indices. (For the unit circle, the singular Cauchy integral appearing above is basically the standard Hilbert transform operator.)

Thus it is clear that Theorem 2.1 gives an elegant generalization of this positive/negative Fourier-parts decomposition of functions on the circle to the situation of an arbitrary rectifiable Jordan curve. This point of view is useful for our work in this article.

Cech cohomology interpretation. That a decomposition as above must exist for fairly arbitrary functions on any Jordan curve can be seen by interpreting things in Cech cohomology of the Riemann sphere $\mathbf{P}^1$ with coefficients in the sheaf of germs of holomorphic functions, (i.e., the structure sheaf $\mathscr{O}$). This is a remark to me by Simha.

In fact, one knows that $H^1(\mathbf{P}^1, \mathscr{O}) = 0$. Let us work with any covering of $\mathbf{P}^1$ by two open sets U_1 and U_2, which are open neighbourhoods (respectively) of the closures of D and $D^\star$. Thus their intersection is some "thin" neighbourhood of the curve γ.

Suppose that f has a complex analytic extension to some arbitrarily thin such annular neighbourhood of γ. Then f determines, by definition, an element of $H^1(\{U_1, U_2\}, \mathscr{O})$. But at the first cohomology level, the map induced by refinements of covering is always *injective*. Hence, since the first cohomology with coefficients in the structure sheaf vanishes, the above cohomology element must be a coboundary. That is exactly the same as saying that *every holomorphic function in any arbitrarily thin annular neighbourhood of any Jordan curve γ can be written as the difference $f = H^+ - H^-$, with H^+ holomorphic on the closure of D and H^- holomorphic on the closure of $D^\star$.*

Now, any real-analytic (complex valued) function on a real-analytic curve will naturally allow holomorphic extension to such an annular neighbourhood. In our situation we only want f to be the difference of traces of holomorphic functions H^+ and H^-, so that an approximation argument by real-analytic objects shows that, under rather mild conditions, "any" function f on a regular curve γ allows a plus/minus parts decomposition of the required type. (The optimal conditions for this matter are not relevant to us at present.)

The upshot is that we have here a nice theoretical justification for the existence of the desired decomposition for functions on curves.

Our aim is to apply this plus/minus-parts decomposition method to the problem of relating conformal welding with the Riemann mapping functions.

3. One-parameter families of Jordan curves

Let γ_t denote a one-parameter family of Jordan curves depending real analytically on the parameter t ($t \in (-\varepsilon, \varepsilon)$). The initial (base) curve γ_0, corresponding to $t = 0$, may be assumed to be even real-analytic. We herein take the attitude that all the information for γ_0 is known, and thence try to determine recursive formulae for calculating the information for each γ_t to arbitrary order in t.

Notations. Let D_t and $D_t^\star$ be the Jordan regions on $\mathbf{P}^1$ separated by γ_t, and F_t and G_t denote the Riemann maps of Δ and $\Delta^\star$ to these regions (respectively). We assume that the Riemann maps are normalized so that the welding homeomorphism for γ_t:

$$\omega_t = {F_t}^{-1} \circ G_t \tag{3.1}$$

becomes a self-homeomorphism of S^1 fixing three points—say $1, -1, i$.

Let the t-expansion for ω_t starting from ω_0 be the following:

$$\omega_t = \omega_0 + t v_1 + t^2 v_2 + t^3 v_3 + \cdots \tag{3.2}$$

where the v_j are some complex functions on S^1.

Let us set up the t-expansions for the Riemann mappings:

$$F_t = F_0 + t\mu_1 + t^2\mu_2 + \cdots \tag{3.3}$$

$$G_t = G_0 + t\nu_1 + t^2\nu_2 + \cdots \tag{3.4}$$

Our main result will exhibit the t-expansions for F_t and G_t assuming that for ω_t to be given; namely, we shall show how to to determine the μ_j and the ν_j from the v_j. The formula for the first variation term (i.e., $j = 1$) has been obtained earlier by Kirillov [K].

The fundamental equation is that *on the unit circle* S^1, one has:

$$F_t \circ \omega_t = G_t. \tag{3.5}$$

Working to first order in t, (neglecting terms $o(t)$), one obtains from this (denoting d/dz by prime):

$$(3.6) \qquad (F_0' \circ \omega_0)v_1 + \mu_1 \circ \omega_0 = \nu_1$$

valid on S^1. Transfer this relationship over to the reference Jordan curve γ_0 by precomposing both sides with the inverse of G_0:

$$(3.7) \qquad (F_0' \circ {F_0}^{-1})(v_1 \circ {G_0}^{-1}) = (\nu_1 \circ {G_0}^{-1}) - (\mu_1 \circ {F_0}^{-1})$$

valid on γ_0. But the first term on the right hand side of (3.7) is holomorphic on the exterior of γ_0, and the second term is holomorphic in the interior of γ_0. We are therefore in the situation of Theorem 2.1 quoted above, and we see that we have proved:

Proposition 3.1. *Given the first variation term v_1 of the family of weldings, set up the function: $\phi_1 = (F_0' \circ {F_0}^{-1})(v_1 \circ {G_0}^{-1})$ on the initial curve γ_0. Applying the plus/minus parts decomposition of Theorem 2.1 to ϕ_1 produces the first variation terms for the Riemann mappings via the explicit formulae:*

$$(3.8+) \qquad \mu_1 = -({\phi_1}^+ \circ F_0)$$

$$(3.8-) \qquad \nu_1 = -({\phi_1}^- \circ G_0).$$

It is now possible to carry through the above analysis *to any order in* t, and get a corresponding function "ϕ_j" on the curve γ_0 whose plus/minus parts decomposition determines μ_j and ν_j. In fact, next, if we neglect terms that are $o(t^2)$, then the welding equation (3.5) generates the following relation on S^1:

$$(F_0' \circ \omega_0)v_2 + \tfrac{1}{2}(F_0'' \circ \omega_0)v_1^2 + (\mu_1' \circ \omega_0)v_1 = \nu_2 - \mu_2 \circ \omega_0.$$

Again we precompose all terms by G_0^{-1}, and set up the function ϕ_2 on the Jordan curve γ_0:

$$(3.9) \quad \phi_2 = (F_0' \circ {F_0}^{-1})(v_2 \circ {G_0}^{-1}) + \tfrac{1}{2}(F_0'' \circ {F_0}^{-1})(v_1^2 \circ {G_0}^{-1}) + (\mu_1' \circ {F_0}^{-1})(v_1 \circ {G_0}^{-1}).$$

This function is *known* on the initial curve since v_1 and v_2 are supplied to us, and we have already determined the holomorphic function μ_1 on the unit disc by the first order formula (3.8+) above. [Of course, each μ_k throughout Δ is determined by the Cauchy formula from its boundary values on S^1. A similar remark is valid for the ν_k.] Hence, applying the Theorem 2.1 decomposition to ϕ_2 on the initial curve γ_0 gives us:

$$(3.10+) \qquad \mu_2 = -({\phi_2}^+ \circ F_0)$$

$$(3.10-) \qquad \nu_2 = -({\phi_2}^- \circ G_0).$$

An elementary induction argument now proves that there are formulae as above for all the μ_k and ν_k:

Theorem 3.2. *Expanding out the welding equation* (3.5) *up to terms of order* t^k, *and passing over to* γ_0 *via* G_0^{-1}, *one obtains on* γ_0 *a relation of the form*:

$$\phi_k[v_1, v_2, \dots, v_k] = (\nu_k \circ G_0{}^{-1}) - (\mu_k \circ F_0{}^{-1}) \tag{3.11}$$

with an explicit universal formula for the function ϕ_k *as a polynomial in the* $(v_j \circ G_0{}^{-1})$, $j = 1, \dots, k$. *Applying Theorem* 2.1 *to this* ϕ_k *we determine as desired*:

$$\mu_k = -(\phi_k{}^{+} \circ F_0) \tag{3.$k+$}$$

$$\nu_k = -(\phi_k{}^{-} \circ G_0). \tag{3.$k-$}$$

Note that the only term in ϕ_k *involving the* k *th variation term* (v_k) *of the weldings is* $(F_0' \circ F_0{}^{-1})(v_k \circ G_0{}^{-1})$.

Remark. Of course, the coefficients of the polynomial ϕ_k involve the μ_1 to μ_{k-1}, and their derivatives on the unit circle. But these are assumed to have been determined by knowledge of the preceding ϕ_j from $j = 1, \dots, (k-1)$. We thus have a recursive procedure for solving the basic problem posed.

Remark. The t could be a *complex parameter*, and all our relations would still be valid. We would then be in the situation of the λ-Lemma of [MSS], and the family of Jordan regions D_t would be automatically a holomorphically varying family of quasidiscs.

4. Variation of power series coefficients of F_t and G_t

In the paper [N2] we had pointed out a remarkably simple identity between the Fourier series coefficients for v_1 and the power series coefficients of the Riemann mapping functions. The results of Section 3 allow us not only to reprove that relationship but go much further.

In [N2] we supposed that some vector field $v(e^{i\theta})d/d\theta$ on the circle S^1 defines, up to first order, the flow of the one-parameter family ω_t of conformal weldings. Expand the vector field in Fourier series on S^1:

$$v(e^{i\theta}) = \sum a_k e^{ik\theta}. \tag{4.1}$$

Note that $\bar{a_k} = a_{-k}$, since the vector field is *real* on S^1. Further, because we have to Möbius-normalize the entire set-up, we may assume that all the weldings fix three points on S^1. That means that v must vanish at these three points. (See [N2], and our earlier papers referenced therein.)

For the associated family of domains, let the Riemann mappings for small values of t be:

$$F_t(z) = z + t(d_2 z^2 + d_3 z^3 + \cdots) + o(t) \tag{4.2}$$

valid in the unit disc, and,

$$G_t(\zeta) = \zeta + t(c_1\zeta^{-1} + c_2\zeta^{-2} + \cdots) + o(t) \tag{4.3}$$

valid in the exterior of the unit disc. The coefficients d_k and c_k appearing above are, of course, the t-derivatives (at $t=0$) of the power series coefficients of the F_t and G_t, respectively. [Notice that since we are deforming the *identity* welding homeomorphism by the vector field v, the initial curve is S^1 itself; consequently, F_0 and G_0 are the identity maps on Δ and $\Delta^\star$, respectively.]

Utilising critically the perturbation formula for the solutions of the Beltrami equation, we had shown in Theorem 1 *of* [N2] *that*:

$$c_{k-1} = ia_{-k} = i\bar{a}_k, \qquad k = 2,3,\ldots \tag{4.4}$$

But (4.4) *is easily seen to be a consequence of formulae* (3.8±) *of Section* 3 *above.*

In fact, the term v_1 of formula (3.2) is obtained from the vector field by the relation: $v_1(e^{i\theta}) = ie^{i\theta}v(e^{i\theta})$. We now apply the plus/minus-parts decomposition to this v_1, and that is trivial to do since we are working on S^1 and we have the Fourier series given to us. We immediately get μ_1 and ν_1 from formulae (3.8±), and comparing with the expansions (4.2), (4.3) we derive:

$$c_{k-1} = ia_{-k} = \overline{d_{k+1}}, \qquad k = 2,3,\ldots \tag{4.5}$$

as desired.

The above relationship shows, moreover, that the variations of the power series coefficients for the Riemann mappings to the interior and exterior of γ_t are essentially (modulo a shift of index) *complex conjugates of each other.*

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