Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 21, 1996, 389–398

A NOTE ON GEHRING'S LEMMA

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Abstract. We give a new proof and an extension of Gehring's lemma using the real method of interpolation.

1. Introduction

Let Ω be a domain in R^n and let f be a non-negative locally integrable measurable function, then for any cube $Q \subset \Omega$, with sides parallel to the coordinate axes, we let

$$\fint_Q f(x)\,dx = \frac{1}{|Q|}\int_Q f(x)\,dx.$$

We say that f satisfies a reverse Hölder inequality if for some $p \in (1,\infty)$ there exists $C \geq 1$ such that

$$\left(\fint_Q f(x)^p\,dx\right)^{1/p} \leq C \fint_Q f(x)\,dx \tag{1}$$

for every cube $Q \subset \Omega$, with sides parallel to the coordinate axes.

A well known result in the theory of weighted norm inequalities due to Gehring (cf. [4]) states that under such conditions f satisfies a higher integrability condition, namely there exists $q > p$ such that $f \in L^q_{\mathrm{loc}}(\Omega)$. More precisely, it is known that if f satisfies (1) then, for sufficiently small $\varepsilon > 0$, and $q = p+\varepsilon > p$, we have

$$\left(\fint_Q f(x)^q\,dx\right)^{1/q} \leq c\left(\fint_Q f(x)^p\,dx\right)^{1/p}$$

for every cube $Q \subset \Omega$, with sides parallel to the coordinate axes. There are by now many versions and extensions of this result (cf. [3], [5], [9], [11], [16], [17]). Let us also note that Muckenhoupt, in his celebrated paper [14], also proves a similar result which in effect proves a rearrangement invariance property of A_p weights (cf. [13] for a brief discussion). In this connection let us also recall the well known fact that reverse Hölder conditions can be used to characterize A_∞

1991 Mathematics Subject Classification: Primary 42B25, 46B70.

(the union of the A_p classes of Muckenhoupt). Moreover, in the applications to partial differential equations it is important to have some more flexibility in the formulation of condition (1) (cf. [11], [6], [17]). For example [11] treats weighted versions of (1) as well as conditions of the form

$$\left(\fint_Q f(x)^p\,dx\right)^{1/p} \leq C \fint_{2Q} f(x)\,dx \tag{2}$$

for every cube Q, with sides parallel to the coordinate axes such that $2Q \subset \Omega$.

The proofs of these results are based on real variable arguments related to the Calderón–Zygmund decomposition and the scale structure of L^p-spaces.

A perusal of the arguments suggests the possibility that a more general principle is at work here. In this note we propose to give a new proof and an extension of Gehring's inequality. First we reinterpret reverse Hölder conditions in the setting of general real interpolation scales of spaces. In this set up Gehring's lemma appears as an inverse of the usual reiteration formulae of Holmstedt. Interpolation theory then suggests the appropriate scaling of the general result. Our analysis allows not only to extend Gehring's inequality in several directions (e.g. weak type reverse Hölder inequalities, reverse Hölder conditions in other interpolation scales, etc.) but also provides a simple proof avoiding the use of Stieltjes integrals (cf. [4], [16], [17]).

In what follows, by a cube we shall always mean one that has sides parallel to the coordinate axes, moreover the letter C shall indicate constants that need not be same in different occurrences.

2. Reformulation of Gehring's inequality

In order to describe our results we reformulate (1) in a suitable form. Let us assume from now on that Ω is a fixed open cube, and all the L^p-spaces are based on Ω. Given $x \in \Omega$, let $Q \subset \Omega$ be such that $x \in Q$, then (1) implies that

$$M_p f(x) \leq cMf(x) \tag{3}$$

where $M = M_1$ is the maximal operator of Hardy–Littlewood associated with Ω and

$$M_p f(x) = \sup_{x\in Q\subset\Omega}\left(\fint_Q |f(x)|^p\,dx\right)^{1/p}.$$

Taking rearrangements in (3) we see, in view of the equivalence (cf. [7]),

$$(Mf)^*(t) \approx \frac{1}{t}\int_0^t f^*(s)\,ds, \qquad 0 < t < |\Omega|, \tag{4}$$

that

$$\left\{\frac{1}{t}\int_0^t f^*(s)^p\,ds\right\}^{1/p} \le c\frac{1}{t}\int_0^t f^*(s)\,ds. \tag{5}$$

Remark 1. Since we are not going to have explicit use for them in this note it is appropriate to observe that covering lemmas and Calderón–Zygmund decompositions are used to prove the equivalence (4).

Now, in order to see how the scale structure of the L^p-spaces enters in this formulation let us recall a well known tool to construct real interpolation scales: the K-functional. Let (A_0, A_1) be a pair of Banach spaces which, for convenience, shall assume to be such that $A_1 \subset A_0$. Then, for $f \in A_0$, $t > 0$, we let

$$K(t, f; A_0, A_1) = \inf_{f=f_0+f_1,\ f_i\in A_i}\{\|f_0\|_{A_0} + t\|f_1\|_{A_1}\}.$$

Thus, in view of the well known formulae (cf. [2])

$$K(t, f, L^1, L^\infty) = \int_0^t f^*(s)\,ds$$

and

$$K(t, f, L^p, L^\infty) \approx \left\{\int_0^{t^p} f^*(s)^p\,ds\right\}^{1/p} \tag{6}$$

we see that (5) can be rewritten as

$$K(t, f, L^p, L^\infty) \le ct^{1-p}K(t^p, f, L^1, L^\infty).$$

At this point let us also recall that, using the K method of interpolation, we have the following characterizations, $1 < p < \infty$,

$$L^p = (L^1, L^\infty)_{1/p', p; K}$$

and moreover, if $q > p$

$$L^q = (L^p, L^\infty)_{1-p/q, q; K}. \tag{7}$$

Thus, we are led to consider the validity of the following implication

$$K(t, f, L^p, L^\infty) \le ct^{1-p}K(t^p, f, L^1, L^\infty) \tag{8}$$

implies that there exists

(9) $q > p$ such that $K(t, f, L^q, L^\infty) \le Ct^{1-q/p}K(t^{q/p}, f, L^p, L^\infty)$.

In order to understand the hypothesis (8) we recall that, although (6) can be obtained directly from the definitions, an equivalent computation of the K-functional for the pair of (L^p, L^∞) can be based on the reiteration theorem of Holmstedt [8], which in this context gives

$$K(t,f,L^p,L^\infty) \approx \left\{ \int_0^{t^p} \left(K(s,f,L^1,L^\infty) s^{-1/p'} \right)^p \frac{ds}{s} \right\}^{1/p}.$$

Since for a fixed element f the K-functional, for any pair of spaces, is such that $K(t)/t$ decreases we see that we always have

$$K(t,f,L^p,L^\infty) \geq ct \frac{K(t^p,f,L^1,L^\infty)}{t^p}.$$

Our assumption on f in (8) thus guarantees that we have the equivalence

$$K(t,f,L^p,L^\infty) \approx t \frac{K(t^p,f,L^1,L^\infty)}{t^p}. \tag{10}$$

Likewise, Holmstedt's formula also implies

$$K(t,f,L^q,L^\infty) \approx \left\{ \int_0^{t^{q/p}} \left(K(s,f,L^p,L^\infty) s^{-(1-p/q)} \right)^q \frac{ds}{s} \right\}^{1/q}$$

and the right hand side of (9) takes the form

$$K(t,f,L^q,L^\infty) \approx t \frac{K(t^{q/p},f,L^p,L^\infty)}{t^{q/p}}.$$

Combinig this with (10) we see that the conjectured estimate reads

$$K(t,f,L^p,L^\infty) \approx t \frac{K(t^p,f,L^1,L^\infty)}{t^p}$$

implies the existence of $q > p$ such that

$$K(t,f,L^q,L^\infty) \approx t \frac{K(t^{q/p},f,L^p,L^\infty)}{t^{q/p}}$$

and using (10) once again (now applied on the right hand side of the previous statement) we finally arrive to

$$K(t,f,L^p,L^\infty) \approx t \frac{K(t^p,f,L^1,L^\infty)}{t^p}$$

implies the existence of $q > p$ such that

$$K(t, f, L^q, L^\infty) \approx t\frac{K(t^q, f, L^1, L^\infty)}{t^q}.$$

In this form Gehring's inequality appears as a reverse inequality of a reiteration theorem. What we seek to prove is that the validity of the estimate at one "point" of the scale implies its validity for other nearby "points". These considerations suggest that a proof of a sharp version of Gehring's inequality could be based on the Holmstedt argument, i.e. via rescaling of good decompositions. Moreover, once framed in this context Gehring's lemma becomes a purely interpolation theory result with potentially wider applicability.

Theorem 1. *Let (A_0, A_1) be an ordered pair of Banach spaces (i.e. $A_1 \subset A_0$) and suppose that $f \in A_0$ is such that for some constant $c > 1$, $\theta_0 \in (0,1)$, $1 \le p < \infty$, we have for every $t \in (0,1)$,*

$$K(t, f; A_{\theta_0,p;K}, A_1) \le ct\frac{K(t^{1/(1-\theta_0)}, f; A_0, A_1)}{t^{1/(1-\theta_0)}}. \tag{11}$$

Then, there exists $\theta_1 > \theta_0$, such that for $q \ge p$, $0 < t < 1$, we have

$$K(t, A_{\theta_1,q;K}, A_1) \approx t\frac{K(t^{1/(1-\theta_1)}, f; A_0, A_1)}{t^{1/(1-\theta_1)}}.$$

Remark 2. We remark that it is routine to formulate analogous results using the E method. It was in fact the observation that the estimates in [11] were related to the E method of interpolation that motivated our results.

3. Proof of the generalized Gehring inequality

While the content of Theorem 1 goes deep into the structure of real interpolation scales the underlying estimates are elementary. Indeed, in view of Holmstedt's formula we can reduce the proof of our result to a simple analytic lemma.

Lemma 2. *Let $h\colon [0,1] \to R^+$ be an increasing function such that $h(s)/s$ is decreasing, and for $\theta \in (0,1)$ let $h_\theta(s) = s^{-\theta}h(s)$. Suppose that there exists $\theta_0 \in (0,1)$, $p \ge 1$, $C > 1$, such that for every $t \in (0,1)$ we have*

$$\int_0^t h_{\theta_0}(s)^p \frac{ds}{s} \le Ch_{\theta_0}(t)^p. \tag{12}$$

Then, there exists $1 > \theta_1 > \theta_0$, such that for $q \ge p$, we have

$$\int_0^t h_{\theta_1}(s)^q \frac{ds}{s} \le Ch_{\theta_1}(t)^q.$$

Before proving the lemma let us show how it can be used to prove Theorem 1. Indeed, under the assumptions of Theorem 1 we have

$$K(t^{1-\theta_0}, f, A_{\theta_0,p;K}, A_1)^p \approx \int_0^t \big(s^{-\theta_0} K(s,f,\bar{A})\big)^p \frac{ds}{s} \le C\big(t^{-\theta_0} K(t,f,\bar{A})\big)^p$$

and therefore if we let $h(s) = K(s,f,\bar{A})$ we can apply the previous lemma to conclude that for suitable $\theta_1 \in (0,1)$, $\theta_1 > \theta_0$, and $q \ge p$

$$K(t^{1-\theta_1}, f, A_{\theta_1,q;K}, A_1)^q \approx \int_0^t \big(s^{-\theta_1} K(s,f,\bar{A})\big)^q \frac{ds}{s} \le C\big(t^{-\theta_1} K(t,f,\bar{A})\big)^q$$

as required.

Let us now proceed with the proof of the lemma. As a first step we shall show that there exists a constant $c \in (0,1)$ such that $s^{-c} h_{\theta_0}(s)^p$ behaves like an increasing function: more precisely we shall show that there exists $c \in (0,1)$ and a constant K such that for every $x, y \in (0,1)$, $x < y$, we have $x^{-c} h_{\theta_0}(x)^p \le K y^{-c} h_{\theta_0}(y)^p$. It is easy to see that if (12) holds, then there exists $c \in (0,1)$ such that

$$\frac{c}{t} \le \frac{d}{dt}\bigg(\log \int_0^t h_{\theta_0}(s)^p \frac{ds}{s}\bigg) \tag{13}$$

(cf. [14] and [15] for similar ideas in closely related contexts).

Let $x, y \in (0,1)$, with $x < y$, and integrate (13) from x to y, then we get

$$c \log \frac{y}{x} \le \log \frac{\int_0^y h_{\theta_0}(s)^p s^{-1}\, ds}{\int_0^x h_{\theta_0}(s)^p s^{-1}\, ds}$$

from where it follows readily that

$$x^{-c} \int_0^x h_{\theta_0}(s)^p \frac{ds}{s} \le y^{-c} \int_0^y h_{\theta_0}(s)^p \frac{ds}{s} \le C y^{-c} h_{\theta_0}(y)^p. \tag{14}$$

On the other hand, by the monotonicity assumptions on h, we have

$$\begin{aligned} x^{-c} \int_0^x h_{\theta_0}(s)^p \frac{ds}{s} &= x^{-c} \int_0^x \Big(s^{1-\theta_0}\Big(\frac{h(s)}{s}\Big)\Big)^p \frac{ds}{s} \\ &\ge x^{-c} \Big(\frac{h(x)}{x}\Big)^p \frac{x^{(1-\theta_0)p}}{(1-\theta_0)p} = \frac{x^{-c} h_{\theta_0}(x)^p}{(1-\theta_0)p}. \end{aligned} \tag{15}$$

Combining (14) and (15) we finally obtain

$$x^{-c} h_{\theta_0}(x)^p \le K y^{-c} h_{\theta_0}(y)^p.$$

Let us now set $q = p(1+\varepsilon)$, $\theta_1 = \theta_0 + \alpha$, with $\varepsilon \geq 0$ and where $\alpha \in (0, 1-\theta_0)$ is such that $\alpha < c/p$. Then,

$$h_{\theta_1}(s)^q = h_{\theta_1}(s)^{p(1+\varepsilon)} = h_{\theta_0}(s)^{p(1+\varepsilon)} s^{-c(1+\varepsilon)} s^{\theta_0 p(1+\varepsilon)} s^{-\theta_1 p(1+\varepsilon)} s^{c(1+\varepsilon)}$$

and therefore we get

$$\begin{aligned}\int_0^t h_{\theta_1}(s)^q \,\frac{ds}{s} &= \int_0^t h_{\theta_0}(s)^{p(1+\varepsilon)} s^{-c(1+\varepsilon)} s^{\theta_0 p(1+\varepsilon)} s^{-\theta_1 p(1+\varepsilon)} s^{c(1+\varepsilon)} \,\frac{ds}{s} \\ &\leq K^{(1+\varepsilon)} h_{\theta_0}(t)^{p(1+\varepsilon)} t^{-c(1+\varepsilon)} \int_0^t s^{\theta_0 p(1+\varepsilon)} s^{-\theta_1 p(1+\varepsilon)} s^{c(1+\varepsilon)} \,\frac{ds}{s}.\end{aligned}$$

Now, the exponents in the last integral add up to

$$\theta_0 p(1+\varepsilon) - (\theta_0 + \alpha)p(1+\varepsilon) + c(1+\varepsilon) - 1 = (-\alpha p + c)(1+\varepsilon) - 1$$

and by the choice of α, $-\alpha p + c > 0$. Therefore,

$$\int_0^t h_{\theta_1}(s)^q \,\frac{ds}{s} \leq K' h_{\theta_0}(t)^{p(1+\varepsilon)} t^{-c(1+\varepsilon)} t^{(-\alpha p + c)(1+\varepsilon)} = K' h_{\theta_1}(t)^q$$

as we wished to show. $\square$

Example 3. In the special case of L^p spaces Theorem 1 gives, in particular, that if

$$\left(\fint_Q f(x)^p \,dx \right)^{1/p} \leq C \fint_Q f(x)\,dx$$

for every cube $Q \subset \Omega$, then, for sufficiently small positive ε, $q = p+\varepsilon$, $0 < t < |\Omega|$, we have

$$\left\{ \frac{1}{t} \int_0^t f^*(s)^q \,ds \right\}^{1/q} \leq C \left\{ \frac{1}{t} \int_0^t f^*(s)^p \,ds \right\}^{1/p}. \tag{16}$$

If we fix a cube Q and localize we observe that

$$\left(\fint_Q f(x)^r \,dx \right)^{1/r} = \left\{ \frac{1}{|Q|} \int_0^{|Q|} (f\chi_Q)^{*r}(s) \right\}^{1/r}$$

and thus we see that (16) gives

$$\left(\fint_Q f(x)^q \,dx \right)^{1/q} \leq C \left(\fint_Q f(x)^p \,dx \right)^{1/p}.$$

Example 4. It is clear that the argument above can be also used to treat weighted reverse Hölder conditions as long as we know that the equivalence (4) is valid for the corresponding weighted local maximal functions. In particular this applies to weights that satisfy a doubling condition (cf. [11]).

Remark 3. It is not difficult to formulate the dual of Lemma 2 which can be proven directly or by duality. The corresponding version of Theorem 1 that obtains is thus associated with the Holmstedt formulae

$$K(t,f;A_0,A_{\theta_0,p}) \approx t\left(\int_{t^{1/\theta_1}}^{\infty} \left(s^{-\theta_1}K(s,f;\bar{A})\right)^p \frac{ds}{s}\right)^{1/p}.$$

4. Examples

We conclude by giving explicit calculations in some concrete settings.

4.1. Weighted L^p spaces: p fixed and variable weights. We consider weighted $L^p(w)$ spaces defined by the norms

$$\|f\|_{L^p(w)} = \left\{\int_\Omega |f(x)|^p w(x)\,dx\right\}^{1/p}.$$

Then, as is well known (cf. [2], [10])

$$K\big(t,f;L^p(w_0),L^p(w_1)\big)^p \approx \int_\Omega |f(x)|^p \min\{w_0(x),t^p w_1(x)\}\,dx.$$

In particular,

$$L^p(w_0^{1-\theta}w_1^{\theta}) = \big(L^p(w_0),L^p(w_1)\big)_{\theta,p;K}.$$

Now, f satisfies a Gehring condition in this setting if there exists $\theta\in(0,1)$, $q\geq 1$, and $c>0$ such that

$$\left\{\int_0^{t^{1/(1-\theta)}} \left(s^{-\theta}\left\{\int_\Omega |f(x)|^p \min\{w_0(x),s^p w_1(x)\}\,dx\right\}^{1/p}\right)^q \frac{ds}{s}\right\}^{1/q} \\ \leq ct\,\frac{\left\{\int_\Omega |f(x)|^p \min\{w_0(x),t^{p/(1-\theta)}w_1(x)\}\,dx\right\}^{1/p}}{t^{1/(1-\theta)}}.$$

4.2. Weighted L^p spaces: p varying. We consider pairs $(L^{p_0}_{w_0},L^{p_1}_{w_1})$ where $1\leq p_i\leq\infty, w_i>0$, and

$$\|f\|_{L^p_w} = \left\{\int_\Omega |f(x)w(x)|^p\,dx\right\}^{1/p}.$$

It is convenient to assume that $w_0=1$, (the general computation follows easily once this case is at hand), then (cf. [10])

$$K(t,f;L^p,L^\infty_w) \approx \left\{\int_0^{t^p} \left((fw)^{\sharp(p)}(s)\right)^p ds\right\}^{1/p}$$

where ${}^{\sharp(p)}$ denotes non-increasing rearrangement with respect to the measure $w(x)^{-p}\,dx$.

In this case a typical Gehring condition for an element f reads: there exists $p>1$, $c>0$, such that for every $t>0$,

$$\left\{\int_0^{t^p}\left((fw)^{\sharp(p)}(s)\right)^p ds\right\}^{1/p}\le t^{1-p}\int_0^{t^p}(fw)^{\sharp(1)}(s)ds.$$

Analytic semigroups. Suppose that A is the generator of an analytic semigroup in a Banach space X, and let D_A denote the domain of A, then it is well known (cf. [2], [13]) that

$$K(t,f;X,D_A)\approx\left\|AR\Big(\frac{1}{t}\Big)f\right\|_X$$

where $R(t)=(A+tI)^{-1}$.

An element f satisfies a Gehring condition if for some $\theta\in(0,1)$, $p\ge 1$, there exists a constant $c>0$ such that we have

$$\left\{\int_0^{t^{1/(1-\theta)}}\left(s^{-\theta}\left\|AR\Big(\frac{1}{t}\Big)f\right\|_X\right)^p\frac{ds}{s}\right\}^{1/p}\le ct\frac{\|AR(t^{-1/(1-\theta)})f\|_X}{t^{1/(1-\theta)}}.$$

Sobolev spaces. We consider the pair of Sobolev spaces $\big(W_p^k(\Omega),W_\infty^k(\Omega)\big)$, $p\ge 1$, where Ω is a smooth domain. It is known (cf. [1] and [13]) that

$$K\big(t,f;W_p^k(\Omega),W_\infty^k(\Omega)\big)\approx\sum_{|\alpha|\le k}\left\{\int_0^{t^p}\left((D^\alpha f)^*(s)\right)^p ds\right\}^{1/p}.$$

Therefore a typical Gehring condition for an element f with respect to the pair $\big(W_1^k(\Omega),W_\infty^k(\Omega)\big)$ reads: there exists $p>1$, $c>0$, such that

$$\sum_{|\alpha|\le k}\left\{\int_0^{t^p}\left((D^\alpha f)^*(s)\right)^p ds\right\}^{1/p}\le ct^{1-p}\sum_{|\alpha|\le k}\int_0^{t^p}(D^\alpha f)^*(s)\,ds.$$

Of course there are weak type variants, the formulation of which we leave to the interested reader.

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Received 15 May 1995