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THE COMPLEMENT OF A QUASIMÖBIUS SPHERE IS UNIFORM

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Abstract. We show that if F is a θ-quasimöbius map of $\mathbf{S}^n$ into $\overline{\mathbf{R}^{n+1}}$, then the complementary components of $F(\mathbf{S}^n)$ are $c(\theta)$-uniform domains. In particular, if F is M bi-Lipschitz, then the complementary components are $c(M)$-uniform domains.

Introduction

Let G be a homeomorphism from $\mathbf{S}^n$ into $\overline{\mathbf{R}^{n+1}}$ where $n \geq 1$. The Jordan–Brouwer separation theorem says that $G(\mathbf{S}^n)$ has two complementary components and that $G(\mathbf{S}^n)$ is the boundary of each. We would like to understand what the complementary components look like when we impose some condition on G that ensures it does not distort too much. For example, G could be bi-Lipschitz. This restriction does not improve the topological nature of the complementary components in that it is possible to have bi-Lipschitz embeddings of the sphere that are wild, that is, at least one of the complementary components is not a topological ball. For example, both the Alexander horned sphere and the boundary of the fattened Artin–Fox wild arc are bi-Lipschitz images of $\mathbf{S}^2$ in $\overline{\mathbf{R}^3}$ (see [G] for the latter). However, we might expect the complementary components not only to be connected in the topological sense, but also to satisfy some metric version of connectivity, such as being uniform. This type of problem is what J. Väisälä calls metric duality, that is, how metric properties of a compact set (such as $G(\mathbf{S}^n)$) affect metric properties of its complement and vice versa. In [V4], he develops a theory of metric duality and proves a theorem that is similar in spirit to the Alexander duality theorem. As a consequence of that theorem, he proves the following result:

Theorem (Väisälä). *If G is a θ-quasimöbius map of $\mathbf{S}^n$ into $\overline{\mathbf{R}^{n+1}}$, then the complementary components of $G(\mathbf{S}^n)$ are c'-uniform domains with $c' = c'(n,\theta,c)$.*

Quasimöbius maps are a useful, Möbius-invariant class of maps introduced by Väisälä in [V1]. They include both the bi-Lipschitz and the quasisymmetric maps.

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The theorem above is true more generally. It holds when $\mathbf{S}^n$ is replaced by a compact set in $\overline{\mathbf{R}^{n+1}}$ each of whose complementary components is c-uniform, homologically trivial, and satisfies a certain higher-dimensional version of c-uniformity (see [V4] for the details). Väisälä has conjectured that even in this general case the constant c' appearing in the theorem is independent of the dimension n (see [V3], [V4]). We will prove this conjecture in the case of $\mathbf{S}^n$.

Theorem X. *If G is a θ-quasimöbius map of $\mathbf{S}^n$ into $\overline{\mathbf{R}^{n+1}}$, then the complementary components of $G(\mathbf{S}^n)$ are $c_0(\theta)$-uniform domains where $c_0(\theta) = 322\,\theta(8)$.*

Every M bi-Lipschitz map is a θ-quasimöbius map with $\theta(t) = M^4 t$. Thus, we obtain:

Corollary. *If G is an M bi-Lipschitz map of $\mathbf{S}^n$ into $\overline{\mathbf{R}^{n+1}}$, then the complementary components of $G(\mathbf{S}^n)$ are $(8M)^4$-uniform domains.*

What follows is a sketch of the proof of Theorem X. We have to show that any two points a, b in the same complementary component of $G(S^n)$ can be joined by a cigar shaped domain, having a and b as end points, that does not intersect $G(S^n)$. In Section 1, we give all the definitions and show that we can assume that $a = \infty$. In Section 2, we use dyadic cubes to construct a set $\mathscr{W}_\infty$ that contains ∞, and whose boundary is close to $G(\mathbf{S}^n)$. We then show that if $b \in \mathscr{W}_\infty$, then there is a cigar joining b and ∞ outside $G(\mathbf{S}^n)$. We show in Section 3 that if $b \notin \mathscr{W}_\infty$ then $\partial\mathscr{W}_\infty$, viewed as an n-cycle, is non-trivial in the complement of $\{\infty, b\}$. However, Section 4 shows that there is a homotopy that moves $\partial\mathscr{W}_\infty$ into $G(\mathbf{S}^n)$ without passing through ∞ or b. This implies that $\partial\mathscr{W}_\infty$ is trivial in the complement of $\{\infty, b\}$. This contradiction proves that $b \in \mathscr{W}_\infty$, and we are done.

1. Preliminaries

We will use $c\,(a_1, \dots, a_m)$ to denote a constant that depends only on $a_1, \dots,$ a_m, and is at least 1. The extended space $\mathbf{R}^{n+1} \cup \{\infty\}$, and the unit n-sphere will be denoted by $\overline{\mathbf{R}^{n+1}}$ and $\mathbf{S}^n$, respectively. We will write $\mathrm{d}(A, B)$ for the distance between two sets A and B.

If a, b, c, d are in $\overline{\mathbf{R}^{n+1}}$ with a and d both being different from b and c, we define their *cross-ratio* to be

$$|a, b, c, d| = \frac{|a-b|\,|c-d|}{|a-c|\,|b-d|},$$

with the obvious modification when one of the points is ∞. The cross-ratio is a positive, finite number and it is Möbius-invariant.

For $M \geq 1$, we say that G is M *bi-Lipschitz* if

$$M^{-1} \leq \frac{|G(a) - G(b)|}{|a-b|} \leq M$$

for any two distinct points a and b in the domain of G.

Our discussion of quasimöbius maps is essentially taken from [V1]. Suppose that W is a subset of $\overline{\mathbf{R}^{n+1}}$ that contains at least two distinct points, and that $f\colon W \to \overline{\mathbf{R}^{n+1}}$ is an embedding. Whenever $\tau = |a,b,c,d|$ is a cross-ratio of points in W, τ' will denote the cross-ratio $|f(a),f(b),f(c),f(d)|$. We say that f is θ-*quasimöbius* if $\theta\colon [0,\infty) \to [0,\infty)$ is a homeomorphism, and $\tau' \le \theta(\tau)$ whenever τ is a cross-ratio of points in W. Note that by choosing $a = d$ and $b = c$, we have $\tau' = \tau = 1$; thus $1 \le \theta(1)$. Because $|a,b,c,d| = |a,c,b,d|^{-1}$, we get the double inequality

$$\frac{1}{\theta(\tau^{-1})} \le \tau' \le \theta(\tau).$$

It is easy to deduce from the left-hand inequality that f^{-1} is θ'-quasimöbius in $f(W)$ with $\theta'(t) = 1/\theta^{-1}(t^{-1})$. If σ is a Möbius map, and f is θ-quasimöbius, then both $\sigma \circ f$ and $f \circ \sigma$ are θ-quasimöbius. If f is M bi-Lipschitz, then it is θ-quasimöbius with $\theta(t) = M^4 t$. Also, if W is a uniform domain in $\overline{\mathbf{R}^{n+1}}$, such as $\overline{\mathbf{R}^{n+1}}$ itself, then f being quasimöbius in W is equivalent to f being quasiconformal in W (Theorem 5.6, [V1]).

A cautionary note: a variety of cross-ratios are used in the literature. For example, quasimöbius maps are defined in [V1] using a different cross-ratio from ours. By relabeling the points, however, one easily sees that the definitions are equivalent.

We will use the distortion estimates of the next lemma in Section 4. A map F satisfying the hypotheses of the lemma is actually $\theta(\eta)$-quasisymmetric (Theorem 3.12, [V1]). This is a stronger conclusion than that of the lemma, except that the lemma gives us explicit constants for the estimates we need.

Lemma 1.1. *Let X and Y be bounded subsets of $\mathbf{R}^{n+1}$ and let F be an η-quasimöbius map from X into Y. Suppose that there are elements x_1, x_2, and x_3 of X satisfying*

$$|x_i - x_j| \ge \tfrac{1}{2}\operatorname{diam} X \qquad \text{and} \qquad |F(x_i) - F(x_j)| \ge \tfrac{1}{2}\operatorname{diam} Y, \ \text{when } i \ne j.$$

We then have the following estimates:

(i) $\dfrac{|F(p)-F(q)|}{\operatorname{diam} Y} \le 2\,\eta\Big(\dfrac{16|p-q|}{\operatorname{diam} X}\Big)$, *for any* $p, q \in X$.

(ii) *If* $|p-q| \le |p-r|$, *then* $|F(p)-F(q)| \le 4\eta(4)\,|F(p)-F(r)|$.

Proof. There is no loss of generality if we assume that $\operatorname{diam} X = \operatorname{diam} Y = 1$. If $p = q$, then (i) and (ii) are trivial. Henceforth we assume that $p \ne q$.

Due to the spacing of the x_i's, if x is any element of X, then at least two of the x_i's are at least a distance $\frac{1}{4}$ from x. Using this fact for p and q, we find that we can choose two of the x_i's, say x_1 and x_2, so that $|p-x_1|, |q-x_2| \ge \frac{1}{4}$. Thus $|p,q,x_1,x_2| \le 16|p-q|$, and consequently $|F(p),F(q),F(x_1),F(x_2)| \le \eta(16|p-q|)$. Simple manipulation of this inequality gives (i).

To prove (ii), assume that p, q, r are elements of X with $|p-q| \le |p-r|$. At least two of the x_i's are at least a distance $\frac{1}{4}$ from q, and at least two of the $F(x_i)$'s are at least a distance $\frac{1}{4}$ from $F(r)$. It follows that we can choose one of the x_i's, say x_1, so that $|q-x_1|, |F(r)-F(x_1)| \ge \frac{1}{4}$. We now have

$$|p,q,r,x_1| = \frac{|p-q||r-x_1|}{|p-r||q-x_1|} \le \Big(\frac{|p-q|}{|p-r|}\Big)\frac{\operatorname{diam} X}{|q-x_1|} \le 4.$$

F being quasimöbius now implies that $|F(p),F(q),F(r),F(x_1)| \le \eta(4)$. From this, (ii) follows easily. □

There are many ways to define uniform domains. For a discussion, see [V2]. For our purposes, we will use the Möbius-invariant definition given in [V2]. This idea is due to Martio [M]. Suppose that K is a continuum in $\overline{\mathbf{R}^{n+1}}$ that contains the distinct points a and b. Let γ denote the triple (K,a,b). For $0 < r \le 1$, we make the following definitions:

$$\begin{aligned}
\operatorname{cig}_1(\gamma,r) &= \{x \in \overline{\mathbf{R}^{n+1}} : |x,y,a,b| < r, \text{ for some } y \in K\},\\
\operatorname{cig}_2(\gamma,r) &= \{x \in \overline{\mathbf{R}^{n+1}} : |x,y,b,a| < r, \text{ for some } y \in K\},\\
\operatorname{cig}(\gamma,r) &= \operatorname{cig}_1(\gamma,r) \cup \operatorname{cig}_2(\gamma,r).
\end{aligned}$$

The idea is that $\operatorname{cig}(\gamma,r)$ is shaped like a cigar, with K as its core and a and b as its ends. It is wide in the middle and narrow at the ends. We will refer to $\operatorname{cig}(\gamma,r)$ as an r-cigar joining a and b. If $a=\infty$, and K is an infinite ray from b, then $\operatorname{cig}(\gamma,r)$ is a cone with vertex b, with K as the central axis, and with aperture $2\sin^{-1} r$. For $c \ge 1$, a domain Ω is said to be a *c-uniform domain* if for each pair of points $a,b \in \Omega$, there is a c^{-1}-cigar joining a and b that lies in Ω. Thus, proving Theorem X means showing the following:

(1.1) If G is a θ-quasimöbius map from $\mathbf{S}^n$ into $\overline{\mathbf{R}^{n+1}}$, and a and b are two points in the same complementary component of $G(\mathbf{S}^n)$, then there is a $c_0(\theta)$ cigar joining a and b that does not intersect $G(\mathbf{S}^n)$.

It will be very useful to make a number of reductions to the problem. Henceforth, M will denote $G(\mathbf{S}^n)$. Take τ_1 to be any Möbius map with $\tau_1(a)=\infty$. The map $\tau_1 \circ G$ is still θ-quasimöbius, and cigars are preserved by Möbius maps. Thus, in (1.1) we can assume that $a=\infty$. Note that this implies that M is bounded. Now choose $z_1,z_2,z_3 \in \mathsf{M}$, with $|z_i - z_j| \ge \frac{1}{2}(\operatorname{diam} \mathsf{M})$ whenever $i \ne j$, and let $x_i = G^{-1}(z_i)$. There is a Möbius map σ that maps $\mathbf{S}^n$ onto itself and maps e_i, the unit vector in the i^{th} direction, to x_i (note: when $n=2$ we take e_3 to be $-e_1$). The map $G\circ\sigma$ is a θ-quasimöbius map from $\mathbf{S}^n$ onto M that maps e_i to x_i for every $1 \le i \le 3$. This allows us to assume that $|G(e_i) - G(e_j)| \ge \frac{1}{2}(\operatorname{diam} \mathsf{M})$, when $1 \le i < j \le 3$. Now let τ_2 be a similarity with the property that $0 \in \tau_2(\mathsf{M})$ and the diameter of $\tau_2(\mathsf{M})$ is $\frac{1}{4}$. Consideration of the map $\tau_2 \circ G$ allows us to further reduce the proof of (1.1) to the case where $0 \in \mathsf{M}$ and the diameter of M is $\frac{1}{4}$.

In summary, we have reduced the proof of Theorem X to proving the following proposition:

Proposition 1.2. *Suppose that G is a θ-quasimöbius map from $\mathbf{S}^n$ into $\overline{\mathbf{R}^{n+1}}$, that b and ∞ lie in the same complementary component of $G(\mathbf{S}^n) = \mathsf{M}$, and that the following two conditions hold*:

(i) $|G(e_i) - G(e_j)| \geq \frac{1}{2}(\operatorname{diam} \mathsf{M})$, *when* $1 \leq i < j \leq 3$.
(ii) $0 \in \mathsf{M}$ *and* $\operatorname{diam} \mathsf{M}$ *is* $\frac{1}{4}$.

Then there is a $c_0(\theta)^{-1}$-cigar joining b and ∞ that does not intersect M.

2. Cubes

Henceforth we assume that G satisfies the hypotheses of Proposition 1.2.

We will write E for $\overline{\mathbf{R}^{n+1}} \setminus \{\infty, b\}$, and δ for the distance from b to M. If $\delta > \frac{1}{4}$, then there is an entire half-plane joining b and ∞. Thus we only need to consider the case where $\delta \leq \frac{1}{4}$.

Take Q_0 to be the cube centred at 0, with sidelength 1. By a cube, we mean a closed cube. Recall that a cube in $\mathbf{R}^{n+1}$ of sidelength λ has diameter $\lambda\sqrt{n+1}$. Let $\beta = 320\,\theta(8)$. The reason for this choice will become clear later. The three estimates we will need are $10 \leq \beta$, $4\,\theta'(320\beta^{-1}) \leq \frac{1}{2}$, and $92\,\theta(4)\,\beta^{-1} < 1$, where θ' is as defined on p. 3.

Choose k_0 so that

$$\tfrac{1}{2}\beta^{-1}\delta < 2^{-k_0}\sqrt{n+1} \leq \beta^{-1}\delta.$$

$\mathscr{D}$ will denote the family of dyadic cubes of sidelength 2^{-k_0}. The diameter of any element of $\mathscr{D}$ lies between $\beta^{-1}\delta/2$ and $\beta^{-1}\delta$. Note that Q_0 is a union of elements of $\mathscr{D}$. Let

$$\begin{aligned}
\mathscr{C} &= \{Q \in \mathscr{D} \text{ for which } \operatorname{diam}(Q) \leq \mathrm{d}(Q,\mathsf{M}) \text{ and } \beta^{-1}\mathrm{d}(Q,b) \leq \mathrm{d}(Q,\mathsf{M})\},\\
\mathscr{W} &= \{\infty\} \cup \bigcup_{Q\in\mathscr{C}} Q.
\end{aligned}$$

We will use the notation Σ^c to denote the complement of a set Σ.

Proposition 2.1. *Any element of $\mathscr{D}$ that intersects $\partial Q_0 \cup Q_0^c$ is an element of $\mathscr{C}$.*

Proof. We will write A for $\partial Q_0 \cup Q_0^c$. Let Q be an element of $\mathscr{D}$ that intersects A and choose $z \in A \cap Q$. We have $\operatorname{diam}(Q) \leq \beta^{-1}\delta \leq 1/40$. Because $z \in A$, we get that $1/4 \leq \mathrm{d}(z,\mathsf{M})$. The triangle inequality gives us $\mathrm{d}(Q,\mathsf{M}) \geq \mathrm{d}(z,\mathsf{M}) - \operatorname{diam}(Q)$. Combining these estimates, we find that $\operatorname{diam}(Q) \leq \mathrm{d}(Q,\mathsf{M})$.

It remains to prove the second estimate in the definition of $\mathscr{C}$. Another consequence of the inequalities in the previous paragraph is that $\mathrm{d}(Q, \mathsf{M}) \geq 9\,\mathrm{d}(z, \mathsf{M})/10$. Making use of this, we obtain

$$\begin{aligned}\mathrm{d}(Q,b) &\leq \mathrm{d}(z,b) \leq \mathrm{d}(z,\mathsf{M}) + \operatorname{diam}(\mathsf{M}) + \mathrm{d}(b,\mathsf{M}) \\ &\leq \tfrac{1}{2} + \mathrm{d}(z,\mathsf{M}) \leq 3\mathrm{d}(z,\mathsf{M}) \leq 4\mathrm{d}(Q,\mathsf{M}) \leq \beta\mathrm{d}(Q,\mathsf{M}). \quad \square\end{aligned}$$

The proposition implies that Q_0^c is contained in $\mathscr{W}$. Because $\mathscr{W}$ is a union of cubes of $\mathscr{D}$, there is no distinction between components and path components, and every component is also a union of cubes of $\mathscr{D}$. We will refer to the unbounded component of $\mathscr{W}$ as $\mathscr{W}_\infty$. Let $\mathscr{C}^*$ be those elements of $\mathscr{C}$ that lie in $\mathscr{W}_\infty$, and that are also contained in Q_0. Define

$$\mathscr{W}^* = \bigcup_{Q \in \mathscr{C}^*} Q.$$

Corollary 2.2.
(a) ∂Q_0 *lies in the interior of* $\mathscr{W}_\infty$.
(b) $\partial \mathscr{W}$, *and hence* $\partial \mathscr{W}_\infty$, *lies in the interior of* Q_0.
(c) $\mathscr{W}^* = Q_0 \cap \mathscr{W}_\infty$.
(d) $\partial \mathscr{W}^* = \partial Q_0 \cup \partial \mathscr{W}_\infty$.

Proof. Parts (a) and (b) follow immediately from the proposition. The inclusion $\mathscr{W}^* \subseteq Q_0 \cap \mathscr{W}_\infty$ is a trivial consequence of the definitions, while the other direction follows from the proposition. Part (d) is a consequence of (a), (b) and (c). $\square$

Lemma 2.3. *If* $b \in \mathscr{W}_\infty$, *then the conclusion of Proposition* 1.2 *holds.*

Proof. Let γ be a path in $\mathscr{W}_\infty$ joining b and ∞, and consider $y \in \gamma$. Some element Q of $\mathscr{C}$ contains y. We then have $|y-b| \leq \operatorname{diam}(Q) + \mathrm{d}(Q,b)$, and $\mathrm{d}(Q,\mathsf{M}) \leq \mathrm{d}(y,\mathsf{M})$. Combining these with the definition of $\mathscr{C}$, we see that

$$(1+\beta)^{-1}|y-b| \leq \mathrm{d}(y,\mathsf{M}).$$

Consequently,

$$\operatorname{cig}_1\bigl(\gamma,(1+\beta)^{-1}\bigr) = \Bigl\{x : \frac{|x-y|}{|y-b|} < (1+\beta)^{-1}\Bigr\}$$

does not intersect M. Lemma 2.8 of [V2] implies that

$$\operatorname{cig}_2\bigl(\gamma,(2+\beta)^{-1}\bigr) = \operatorname{cig}_2\Bigl(\gamma, \frac{(1+\beta)^{-1}}{1+(1+\beta)^{-1}}\Bigr) \subseteq \operatorname{cig}_1\bigl(\gamma,(1+\beta)^{-1}\bigr).$$

We now have $\operatorname{cig}\bigl(\gamma,(2+\beta)^{-1}\bigr) \subseteq \operatorname{cig}_1\bigl(\gamma,(1+\beta)^{-1}\bigr)$, and so $\operatorname{cig}\bigl(\gamma,(2+\beta)^{-1}\bigr)$ does not intersect M. $\square$

In the previous section, we reduced the proof of Theorem X to proving Proposition 1.2. The preceding lemma further reduces the proof to showing that b lies in $\mathscr{W}_\infty$. The remainder of the paper is devoted to showing this.

3. Some topology

Throughout this section we will assume that $b \notin \mathscr{W}_\infty$.

For $k \geq 0$, $A_k(X)$ will denote the free Abelian group generated by the set of all singular k-cubes in X, and ∂_k will denote the boundary operator from $A_k(X)$ into $A_{k-1}(X)$ (see [Ma] for the definitions). Recall that $\partial_{k-1}\partial_k = 0$. The groups $A_k(\partial Q_0)$ and $A_k(\partial \mathscr{W}_\infty)$ are subgroups of $A_k(E)$.

For every $Q \in \mathscr{D}$ there is a unique dilation δ and a unique translation τ such that $\tau \circ \delta$ maps the unit cube onto Q. This map is an element of $A_{n+1}(E)$ and we denote it by $[Q]$. $\mathscr{W}^*$ is represented in $A_{n+1}(E)$ by

$$[\mathscr{W}^*] = \sum_{Q \in \mathscr{C}^*} [Q].$$

It is easily verified that $\partial_{n+1}[\mathscr{W}^*]$ can be written as a sum of an element of $A_n(\partial Q_0)$ and an element of $A_n(\partial \mathscr{W}_\infty)$. We will write $[\partial Q_0]$ and $[\partial \mathscr{W}_\infty]$ respectively for these elements. We then have

$$0 = \partial_n \partial_{n+1}[\mathscr{W}^*] = \partial_n[\partial Q_0] + \partial_n[\partial \mathscr{W}_\infty]. \tag{3.1}$$

In particular,

$$\partial_n[\partial Q_0] = -\partial_n[\partial \mathscr{W}_\infty].$$

The sets $\partial \mathscr{W}_\infty$ and ∂Q_0 are disjoint; thus $A_{n-1}(\partial Q_0)$ and $A_{n-1}(\partial \mathscr{W}_\infty)$ have only the zero element in common. It now follows from the previous equation that

$$0 = \partial_n[\partial Q_0] = \partial_n[\partial \mathscr{W}_\infty].$$

Consequently, we can view $[\partial Q_0]$ and $[\partial \mathscr{W}_\infty]$ as elements of $H_n(E)$, the n^{th} singular homology group of E. These elements are denoted by $\langle \partial Q_0 \rangle$ and $\langle \partial \mathscr{W}_\infty \rangle$ respectively. Recalling that $\partial_{n+1}[\mathscr{W}^*] = [\partial Q_0] + [\partial \mathscr{W}_\infty]$, we see that $\langle \partial Q_0 \rangle + \langle \partial \mathscr{W}_\infty \rangle = 0$. However, $H_n(E)$ is an infinite cyclic group and the element $\langle \partial Q_0 \rangle$ is a generator of $H_n(E)$, whence $\langle \partial \mathscr{W}_\infty \rangle \neq 0$.

The next lemma will be used to prove by contradiction that $b \in \mathscr{W}_\infty$.

Lemma 3.1. *If $P\colon \partial \mathscr{W}_\infty \to \mathsf{M}$ is a continuous map with the property that the line segment joining each $x \in X$ to $P(x)$ does not contain b, then $\langle \partial \mathscr{W}_\infty \rangle = 0$.*

Proof. Take i to be the injection from $\partial \mathscr{W}_\infty$ into E. Then P and i give rise to the induced homomorphisms $P_*, i_*\colon H_n(\partial \mathscr{W}_\infty) \to H_n(E)$. $\langle \partial \mathscr{W}_\infty \rangle$ lies in the range of i_*.

Define $h\colon \partial \mathscr{W}_\infty \times \mathrm{I} \to E$ by

$$h(x,t) = (1-t)x + tP(x).$$

This is a homotopy between i and P. Theorem II.4.1 of [Ma] implies that $P_* = i_*$. Thus, $\langle \partial \mathscr{W}_\infty \rangle$ lies in the range of P_*.

Because b and ∞ lie in the unbounded component of M^c, there is a path joining them that does not intersect M. In fact, we can find a Jordan arc γ that joins b and ∞ and that does not intersect M. Now $H_n(\overline{\mathbf{R}^{n+1}} \setminus \gamma) = 0$ by Lemma III.6.2 in [Ma]. Let $\widetilde{P}$ be the map P viewed as a map from $\partial \mathscr{W}_\infty$ into $\overline{\mathbf{R}^{n+1}} \setminus \gamma$, and let j be the injection from $\overline{\mathbf{R}^{n+1}} \setminus \gamma$ into E. Then we have $j_* \widetilde{P}_* = P_*$. But $\widetilde{P}_* = 0$, since $H_n(\overline{\mathbf{R}^{n+1}} \setminus \gamma) = 0$. Hence, $P_* = 0$. It follows that $\langle \partial \mathscr{W}_\infty \rangle = 0$. □

4. Building the homotopy

In this section, we construct a map P satisfying the hypotheses of Lemma 3.1. The set $\partial \mathscr{W}_\infty$ is a finite union of n-faces of cubes in $\mathscr{D}$. Let

$$\mathscr{F}_k = \{\text{the } k\text{-faces of the } n\text{-faces that make up } \partial \mathscr{W}_\infty\},$$
$$\mathscr{B} = \{\text{the centres of the elements of each } \mathscr{F}_k,\ \forall\ 0 \le k \le n\},$$
$$\Sigma_k = \mathscr{B} \cup \{\text{the pointwise union of the elements of } \mathscr{F}_k\}.$$

Note that $\Sigma_0 = \mathscr{B}$ and $\Sigma_n = \mathscr{W}_\infty$. Define ρ on Σ_0 by choosing, for each $s \in \Sigma_0$, a nearest point w in M and setting $\rho(s) = w$. Instead of directly extending ρ to a continuous map from $\partial \mathscr{W}_\infty$ into M, we will first extend the map $\tau = G^{-1} \circ \rho\colon \Sigma_0 \to \mathbf{S}^n$ to a continuous map from $\partial \mathscr{W}_\infty$ into $\mathbf{S}^n$. This extension will have the property that if $z \in F$, where $F \in \mathscr{F}_n$, then $\tau(z)$ is relatively close to $\tau(c_F)$, where c_F is the centre of F. The estimates of Lemma 1.1 then guarantee that $G \circ \tau(z)$ is relatively close to $G \circ \tau(c_F) = \rho(c_F)$. We will show that the line segment joining c_F and $\rho(c_F)$ is relatively far from b. Consequently, the line segment joining z and $G \circ \tau(z)$ is relatively far from b. Thus, $G \circ \tau$ is the desired extension. To make all of this precise we need some estimates, of course. These are provided by the following lemma:

Lemma 4.1. *If $Q \in \mathscr{C}$, then*

$$1 \le \frac{\mathrm{d}(z, \mathsf{M})}{\mathrm{d}(Q, \mathsf{M})} \le 2, \qquad \text{for all } z \in Q. \tag{4.1}$$

If $Q \in \mathscr{C}$, and $Q \cap \partial \mathscr{W} \ne \emptyset$, then we have the following estimates:

$$1 \le \frac{\mathrm{d}(z, b)}{\mathrm{d}(Q, b)} \le 2, \quad \text{for all } z \in Q, \tag{4.2}$$

$$1 \le \frac{\beta\, \mathrm{d}(Q, \mathsf{M})}{\mathrm{d}(Q, b)} \le 4, \tag{4.3}$$

$$\operatorname{diam}(Q) \le \mathrm{d}(Q, \mathsf{M}) \le \beta^{-1}. \tag{4.4}$$

Proof. The left-hand inequalities of (4.1) and (4.2) are trivial, while the left-hand inequalities of (4.3) and (4.4) follow from the definition of $\mathscr{C}$. It remains to prove the right-hand inequalities.

Let $Q \in \mathscr{C}$, and $z \in Q$. We have $\mathrm{d}(z, \mathsf{M}) \leq \mathrm{diam}(Q) + \mathrm{d}(Q, \mathsf{M}) \leq 2\,\mathrm{d}(Q, \mathsf{M})$, which gives us (4.1).

Now assume that $Q \in \mathscr{C}$, and that $Q \cap \partial\mathscr{W} \neq \emptyset$. Then one of the two cases below must occur:

Case (i) One of the elements of $\mathscr{D}$ that touches Q, call it Q_1, violates the first condition in the definition of $\mathscr{C}$.

This means that $\mathrm{d}(Q_1, \mathsf{M}) < \mathrm{diam}\,(Q_1)$. Then

$$\mathrm{d}(Q, \mathsf{M}) \leq \mathrm{diam}\,(Q_1) + \mathrm{d}(Q_1, \mathsf{M}) < 2\,\mathrm{diam}\,(Q_1) \leq 2\beta^{-1}\delta. \tag{4.5}$$

This implies (4.4), because $\delta \leq \frac{1}{4}$. Making use of (4.5), we have

$$\delta = \mathrm{d}(b, \mathsf{M}) \leq \mathrm{d}(Q, b) + \mathrm{diam}\,(Q) + \mathrm{d}(Q, \mathsf{M}) \leq \mathrm{d}(Q, b) + 3\beta^{-1}\delta,$$

from which we obtain

$$\mathrm{d}(Q, b) \geq (1 - 3\beta^{-1})\delta \geq \frac{7\delta}{10}, \tag{4.6}$$

since $\beta \geq 10$. Combining (4.5) and (4.6), we get (4.3). A final chain of inequalities yields (4.2):

$$\mathrm{d}(z, b) \leq \mathrm{diam}\,(Q) + \mathrm{d}(Q, b) \leq \beta^{-1}\delta + \mathrm{d}(Q, b) \leq \Big(1 + \frac{\beta^{-1}}{1 - 3\beta^{-1}}\Big)\mathrm{d}(Q, b) \leq 2\,\mathrm{d}(Q, b).$$

Case (ii) All the elements of $\mathscr{D}$ that touch Q satisfy the first condition in the definition of $\mathscr{C}$, but one of them, which we will call Q_2, violates the second condition in the definition of $\mathscr{C}$.

We have $\mathrm{d}(Q_2, \mathsf{M}) < \beta^{-1}\mathrm{d}(Q_2, b)$, and $\mathrm{diam}\,(Q_2) \leq \mathrm{d}(Q_2, \mathsf{M})$. First,

$$\mathrm{d}(Q, \mathsf{M}) \leq \mathrm{diam}\,(Q_2) + \mathrm{d}(Q_2, \mathsf{M}) \leq \beta^{-1}\delta + \beta^{-1}\mathrm{d}(Q_2, b).$$

Next we have

$$\mathrm{d}(Q_2, b) \leq \mathrm{d}(Q_2, \mathsf{M}) + \mathrm{diam}\,(\mathsf{M}) + \mathrm{d}(b, \mathsf{M}) \leq \beta^{-1}\mathrm{d}(Q_2, b) + \tfrac{1}{4} + \delta \leq \beta^{-1}\mathrm{d}(Q_2, b) + \tfrac{1}{2}.$$

Combining this with our previous estimate, we obtain (4.4):

$$\mathrm{d}(Q, \mathsf{M}) \leq \beta^{-1}\delta + \frac{\beta^{-1}}{2(1 - \beta^{-1})} \leq \beta^{-1}.$$

To prove (4.2) and (4.3) we need to get an estimate on $\mathrm{d}(Q,b)$, which we derive thus:

$$\begin{aligned}\tfrac{1}{2}\delta &\le \beta \operatorname{diam}(Q_2) \le \beta \mathrm{d}(Q_2,\mathsf{M}) < \mathrm{d}(Q_2,b)\\ &\le \operatorname{diam}(Q) + \mathrm{d}(Q,b) \le \beta^{-1}\delta + \mathrm{d}(Q,b) \le \delta/10 + \mathrm{d}(Q,b).\end{aligned}$$

Consequently,

$$2\delta/5 \le \mathrm{d}(Q,b). \tag{4.7}$$

We can now prove (4.2):

$$\mathrm{d}(z,b) \le \operatorname{diam}(Q) + \mathrm{d}(Q,b) \le \beta^{-1}\delta + \mathrm{d}(Q,b) \le \bigl(1+\tfrac{5}{2}\beta^{-1}\bigr)\mathrm{d}(Q,b) \le 2\,\mathrm{d}(Q,b).$$

Finally, we prove (4.3):

$$\begin{aligned}\beta\,\mathrm{d}(Q,\mathsf{M}) &\le \beta \operatorname{diam}(Q_2) + \beta\,\mathrm{d}(Q_2,\mathsf{M}) < \beta \operatorname{diam}(Q_2) + \mathrm{d}(Q_2,b)\\ &\le \beta \operatorname{diam}(Q_2) + \operatorname{diam}(Q_2) + \mathrm{d}(Q,b) \le (1+\beta^{-1})\,\delta + \mathrm{d}(Q,b)\\ &\le \tfrac{5}{2}(1+\beta^{-1})\mathrm{d}(Q,b) + \mathrm{d}(Q,b) \le 4\,\mathrm{d}(Q,b). \ \square\end{aligned}$$

Corollary 4.2. *Suppose that $Q \in \mathscr{C}$ and that $Q \cap \partial\mathscr{W}_\infty \neq \emptyset$. If $y,z \in Q\cap\Sigma_0$, then*

$$|\rho(y)-y| \le 2\,\mathrm{d}(Q,\mathsf{M}) \qquad \textit{and} \qquad |\rho(y)-\rho(z)| \le 5\,\mathrm{d}(Q,\mathsf{M}).$$

The corollary is an easy consequence of the definition of ρ, the right-hand inequality of (4.1) and the left-hand inequality of (4.4).

Let $F \in \mathscr{F}_n$. There is a unique $Q \in \mathscr{C}$ that contains F, because $F \subset \partial\mathscr{W}_\infty$. From Corollary 4.2 and (4.4), we see that

$$|\rho(y)-\rho(c_F)| \le 5\,\mathrm{d}(Q,\mathsf{M}) \le 5\beta^{-1} = 20\beta^{-1}\operatorname{diam}(\mathsf{M}), \qquad \text{for all } y \in F\cap\Sigma_0.$$

Using estimate (i) in Lemma 1.1, we deduce that

$$|\tau(y)-\tau(c_F)| = |G^{-1}\circ\rho(y)-G^{-1}\circ\rho(c_F)| \le 2\,\theta'(320\beta^{-1})\ \operatorname{diam}(\mathbf{S}^n), \quad \forall\, y \in F\cap\Sigma_0.$$

Our choice of β implies that the right-hand side is bounded by $\frac{1}{2}$. Choose e_F to be an element of $F\cap\Sigma_0$ that maximizes $|\tau(y)-\tau(c_F)|$, for $y \in F\cap\Sigma_0$. Define $b(F)$ to be the intersection of $\mathbf{S}^n$ and $\overline{B}(\tau(c_F),|\tau(e_F)-\tau(c_F)|)$. Because $|\tau(e_F)-\tau(c_F)|$ is at most $\frac{1}{2}$, we know that there is a unique geodesic of $\mathbf{S}^n$ joining any two points of $b(F)$, and that this geodesic lies in $b(F)$.

The map τ has already been defined on Σ_0, and we are going to extend it to $\partial\mathscr{W}_\infty$ $(=\Sigma_n)$ by defining it inductively on each Σ_k. For convenience, we will denote any of the various extensions by τ also. We will say that $g \in \mathscr{G}$ if g is a continuous map from some subset A of $\mathscr{W}_\infty$ into $\mathbf{S}^n$, and $g(A\cap F)\subseteq b(F)$ for any $F\in\mathscr{F}_n$.

Assume that τ has been extended to a continuous map from Σ_k into $\mathbf{S}^n$, and that $\tau\in\mathscr{G}$. This is certainly true for $k=0$. Take $\Phi\in\mathscr{F}_{k+1}$, c to be the center of Φ, and F to be some element of $\mathscr{F}_n$ containing Φ. Our symbol for the pointwise union of the k-faces of Φ is $\delta_k\Phi$. Now, let $x\in\delta_k\Phi$. The line segment joining c and x will be denoted by $[c,x]$. The properties below are easily verified:

(i) $[c,x]\cap\Sigma_k=\{c,x\}$.
(ii) $\Phi=\bigcup_{x'\in\delta_k\Phi}[c,x']$.
(iii) $[c,x]\cap[c,x']=\{c\}$, when $x\neq x'\in\delta_k\Phi$.
(iv) $\Phi\cap\Sigma_k=\delta_k\Phi\cup\{c\}$.

Because $x\in\Sigma_k$, our inductive assumption implies that $\tau(x)\in b(F)$. By definition, $\tau(c)\in b(F)$. For $0\le t\le 1$, let $\gamma(x,t)$ denote the point on the geodesic joining $\tau(c)$ and $\tau(x)$ whose distance from $\tau(c)$ along the geodesic is t times the length of the geodesic. We define τ, a map from $[c,x]$ into $\mathbf{S}^n$, by $\tau\big((1-t)c+tx\big)=\gamma(x,t)$. Note that $\tau([c,x])\subset b(F)$. Doing this for all $x\in\delta_k\Phi$, and using the facts (i)–(iv) above, we see that we get a well-defined extension of τ to $\Sigma_k\cup\Phi$. This extension is an element of $\mathscr{G}$. We can do this for any element of $\mathscr{F}_{k+1}$. If Φ' is another element of $\mathscr{F}_{k+1}$, we have $\Phi\cap\Phi'=\delta_k\Phi\cap\delta_k\Phi'\subseteq\Sigma_k$. This fact allows us to give a well-defined extension of τ to all of Σ_{k+1}. This extension is also an element of $\mathscr{G}$. It follows by induction that we can extend τ to a map of $\partial\mathscr{W}_\infty$ that is an element of $\mathscr{G}$. In particular, we have $\tau(F)\subseteq b(F)$ for any $F\in\mathscr{F}_n$. We now define P to be $G\circ\tau$. This maps $\partial\mathscr{W}_\infty$ into M, and is a continuous extension of the map ρ defined earlier.

Lemma 4.3. *For any $x\in\partial\mathscr{W}_\infty$, the line segment joining x and $P(x)$ does not contain b.*

Proof. Let $x\in\partial\mathscr{W}_\infty$. Choose $F\in\mathscr{F}_n$ that contains x. We have $\tau(F)\subseteq b(F)$. The definition of $b(F)$ and estimate (ii) of Lemma 1.1 give us the following information:

$$\begin{aligned}|P(x)-P(c_F)|&=\big|G\big(\tau(x)\big)-G\big(\tau(c_F)\big)\big|\le 4\theta(4)\big|G\big(\tau(e_F)\big)-G\big(\tau(c_F)\big)\big|\\&=4\theta(4)|P(e_F)-P(c_F)|=4\theta(4)|\rho(e_F)-\rho(c_F)|.\end{aligned}$$

Denote the unique element of $\mathscr{C}$ that contains F by Q. Recalling the estimates of Lemma 4.1 and Corollary 4.2, we obtain

$$\begin{aligned}|P(x)-x|&\le|P(x)-P(c_F)|+|P(c_F)-c_F|+|c_F-x|\\&\le 4\theta(4)|\rho(e_F)-\rho(c_F)|+|\rho(c_F)-c_F|+|c_F-x|\\&\le 20\theta(4)\mathrm{d}(Q,\mathsf{M})+2\mathrm{d}(Q,\mathsf{M})+\operatorname{diam}(Q)\\&\le 23\theta(4)\mathrm{d}(Q,\mathsf{M})\le 70\,\theta(4)\,\beta^{-1}\mathrm{d}(Q,b)\\&\le 92\theta(4)\beta^{-1}|x-b|<|x-b|.\end{aligned}$$

This completes the proof of the lemma. □

We can now conclude the proof of Theorem X. As mentioned previously, it suffices to show that $b \in \mathscr{W}_\infty$. Assume that this is not so. We saw in Section 3 that this implies that $\langle \partial \mathscr{W}_\infty \rangle \neq 0$. However, Lemma 3.1 and the lemma just proven, together imply that $\langle \partial \mathscr{W}_\infty \rangle = 0$. This contradiction means that b must be an element of $\mathscr{W}_\infty$, and so we are done.

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