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ALGEBRAIC SCHWARZIAN DIFFERENTIAL EQUATIONS

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Abstract. Necessary conditions for the Newton polygon associated with a general differential equation algebraic in the unknown function and its Schwarzian derivative are presented in the case of existence of an admissible solution (in Nevanlinna sense). Moreover, an admissible solution is shown to satisfy a stronger version of Nevanlinna's defect relation.

1. Preliminaries

An algebroid function α is an element of the algebraic closure $\mathscr{A}$ of the field $\mathscr{M}$ of meromorphic functions on $\mathbf{C}$. The degree of α is the degree of its minimal polynomial over $\mathscr{M}$. Finitely many algebroid functions $\alpha^{(i)}$, $i=1,\dots,r$, with minimal polynomials P_i can be associated with meromorphic functions on a Riemann surface X of finite number of sheets. Namely, there are a (holomorphic and proper) covering $\pi_X\colon X\to\mathbf{C}$ (of finite number of sheets) and meromorphic functions $\alpha_X^{(i)}$ on X such that

$$\pi_X^*\,P_i\bigl(\alpha_X^{(i)}\bigr)=0,\qquad i=1,\dots,r.$$

Here, $\pi_X^*\,P_i$ is the polynomial obtained from P_i by replacing each coefficient p by $\pi_X^*p=p\circ\pi_X$. We call $\bigl(\pi_X;\alpha_X^{(1)},\dots,\alpha_X^{(r)}\bigr)$ a (simultaneous) realization of the $\alpha^{(i)}$; less exactly, we will also call the $\alpha_X^{(i)}$ realizations of the $\alpha^{(i)}$. We remark that (π_X,π_X) is also a realization of an algebroid function π. If in (π_X,α_X) the number of sheets of π_X equals the degree of α, the realization will be called minimal. A minimal realization always exists (see [Fo]). The ramification number of α_X in $x\in X$ is the multiplicity of π_X in x reduced by 1, i. e.

$$\operatorname{ram}(x,\alpha_X)=\operatorname{mult}(x,\pi_X)-1.$$

When there is no danger of confusion, we simplify our notation identifying algebroid functions and their realizations and omitting the index X.

Now we recall the definitions of the functionals of Nevanlinna theory for algebroid functions (see [Ul]). Let $\pi\colon X\to\mathbf{C}$ be an arbitrary covering, where the number of sheets is $q<\infty$. If $F\colon X\to[0,\infty]$ is continuous, $F(y)=\infty$, and ζ

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a local coordinate around y, then we define the order of infinity of y to be the infimum of the set

$$\left\{l \in [0,\infty) \mid\ F \circ \zeta^{-1}(z) = O(|z|^{-l}),\ z \to 0\right\} \cup \{\infty\}.$$

Let $\mathscr{C}_X$ be the smallest set of continuous functions from X to $[0,\infty]$ that contains all functions $|f|$ where $f\colon X \to \widehat{\mathbf{C}}$ is meromorphic, and is closed with respect to the operations addition, division, and taking the maximum of two functions (see [Er]). It is obvious that the points of infinity of functions in $\mathscr{C}_X$ have integer orders. These points will also be called poles. We are now ready to introduce the Nevanlinna entities, which we are going to define for $\mathscr{C}_X$. Let $\Gamma_r := \pi^{-1}(\{z \mid\ |z| = r\})$ and B_r be a finite subset of Γ_r such that the restriction of π to each connectivity component of $\Gamma_r' := \Gamma_r \setminus B_r$ is injective. Denote the connectivity components by $\gamma_1, \dots, \gamma_l$, the restriction of π to γ_i by π_i, and define $s_i := \pi(\gamma_i)$. Then the proximity function of $F \in \mathscr{C}_X$ is defined as

$$m(r,F) := \frac{1}{2\pi r q} \sum_{i=1}^{l} \int_{s_i} \overset{+}{\log} F\big(\pi_i^{-1}(z)\big)\, d|z|.$$

It is easy to see that $m(r,F)$ does not depend on the choice of B_r. In order to define counting functions $n(r,F)$, $\overline{n}(r,F)$, and $\overline{n}_1(r,F)$, let $\Delta_r := \pi^{-1}(\{z \mid\ |z| \le r\})$ and denote by $n(r,F)$ and $\overline{n}(r,f)$ the number of poles of F in Δ_r, counting and ignoring multiplicities, respectively, and denote by $\overline{n}_1(r,F)$ the number of multiple poles of F in Δ_r, ignoring multiplicities. Then

$$N(r,F) := \frac{1}{q} \int_0^r \frac{n(t,F) - n(0,F)}{t}\, dt + \frac{1}{q}\, n(0,F) \log r \tag{1.1}$$

and $\overline{N}(r,F)$ and $\overline{N}_1(r,F)$ analogous. As usual, we define the characteristic function to be

$$T(r,F) := m(r,F) + N(r,F).$$

If α is an algebroid function with a realization (π, α_X) we write

$$\begin{aligned} m(r,\alpha) &:= m(r,|\alpha_X|),\\ N(r,\alpha) &:= N(r,|\alpha_X|),\\ T(r,\alpha) &:= T(r,|\alpha_X|). \end{aligned}$$

It is not difficult to prove that these definitions of $m(r,\alpha)$ etc. are independent of the choice of the realization and coincide with the usual definitions. If the realization of α is minimal, we get the counting function $N_{\mathrm{ram}}(r,\alpha)$ of ramification points of α by replacing in (1.1) the pole order of α_X by the ramification number. We assume the reader acquainted with the standard theorems of Nevanlinna theory, such as the fundamental theorems and the lemma on the logarithmic derivative (see [Ul]).

For two functions $\Phi, \Psi\colon (0,\infty) \to \mathbf{R}$, the expression

$$\Phi(r) \approx \Psi(r) \qquad \bigl(\Phi(r) \lesssim \Psi(r)\bigr)$$

means that there is an exceptional set $E \subset (0,\infty)$ of finite linear measure such that $\Phi(r) = \Psi(r)$ (respectively $\Phi(r) \le \Psi(r)$) holds outside E.

Let $\varphi\colon (0,\infty) \to (0,\infty)$ be a function of growth sufficiently fast to guarantee

$$\log r = O\bigl(\varphi(r)\bigr).$$

Then $\mathscr{A}_\varphi$ consists of all algebroid functions γ that fulfill

$$T(r,\gamma) \approx O\bigl(\varphi(r)\bigr).$$

In particular, if $\varphi(r) = \log(r)$, $\mathscr{A}_{\log}$ is the set of algebraic functions.

By the well-known properties of the characteristic functions, it is immediate that $\mathscr{A}_\varphi$ is an algebraically and differentially closed subfield of the field $\mathscr{A}$ of all algebroid functions.

The Schwarzian derivative of a non-constant algebroid function α is

$$S\alpha := \Bigl(\frac{\alpha''}{\alpha'}\Bigr)' - \frac{1}{2}\Bigl(\frac{\alpha''}{\alpha'}\Bigr)^2.$$

The article deals with differential equations having the form

$$P(\alpha, S\alpha) = 0, \tag{1.2}$$

where P is an irreducible polynomial in two unknowns over $\mathscr{A}_\varphi$. A solution α of (1.2) is called admissible if

$$\varphi(r) \approx o\bigl(T(r,\alpha)\bigr) \tag{1.3}$$

and

$$N_{\mathrm{ram}}(r,\alpha) \approx O\bigl(\varphi(r)\bigr). \tag{1.4}$$

In particular, (1.3) holds whenever $\mathscr{A}_\varphi = \mathscr{A}_{\log}$ is the set of algebraic functions and α is transcendental.

We now proceed to provide the algebraic background. Let K be an algebraically closed field of characteristic 0. Later, we will have $K = \mathscr{A}_\varphi$. If x is transcendental over K and y is algebraic over $K(x)$ then $K(x)$ is a rational function field and $L = K(x,y)$ an algebraic function field over K. All rational function fields over K are of course isomorphic. A discrete valuation of L over K is a mapping

$$\nu\colon L \to \mathbf{R} \cup \{\infty\}$$

with the following properties (see [De, II §2]):
(i) $\nu(u) = \infty$ if and only if $u = 0$.
(ii) $\nu(uv) = \nu(u) + \nu(v)$.
(iii) $\nu(u+v) \geq \min\{\nu(u), \nu(v)\}$ (strong triangular inequality).
(iv) $\nu(L^*)$ is isomorphic to $\mathbf{Z}$ as an additive group, where $L^* = L \setminus \{0\}$.
(v) $\nu(a) = 0$ for each $a \in K$.
Since all valuations treated here will be discrete and over K, we will leave these attributes out and merely speak of valuations. ν is said to be normal if $\nu(L^*) = \mathbf{Z}$. The set of all normal valuations of L will be denoted by $\mathscr{N}(L)$.

We remark that some authors, so in [Ca], define valuations in a different, however equivalent, way. Namely, taking a valuation ν and a real number $c \in (0,1)$, the mapping

$$a \mapsto c^{\nu(a)}$$

would satisfy that definition of a valuation.

For a rational function field $K(u)$, $u \in L$, there is a bijection between $K \cup \{\infty\}$ and $\mathscr{N}\big(K(u)\big)$. Namely, for $a \in K$, $\nu_{u,a}$ is defined in the following way: If $w = P(u)/Q(u)$, where the power of $(u-a)$ in $P(u)$ is m and in $Q(u)$ is n, then $\nu_{u,a}(w) = m - n$. For $a = \infty$, $\nu_{u,\infty}(w) = \deg Q - \deg P$. Hence $\nu_{u,a}(w)$ is the formal order of the zero of w in a.

For each polynomial P in two unknowns u and w over K and each $a \in K \cup \{\infty\}$, there is a geometrical construction called the Newton polygon of P with respect to u and a (see [Ca]), and a finite sequence called the Newton type. In order to obtain them, we first write P as a polynomial in w,

$$P = P(u,w) = \sum_{j=0}^{n} P_j(u)\, w^j,$$

where the coefficients $P_j(u)$ are polynomials in u over K, thus elements of $K(u)$. For each j with $P_j \neq 0$, we let $\nu_j := \nu_{u,a}\big(P_j(u)\big)$ and mark the point (j, ν_j) in the two-dimensional coordinate system. Taking the lower part of the convex hull of the generated points, we get the Newton polygon. Say it consists of r edges. The slopes μ_i of the edges are called Newton slopes; if the i-th edge ranges from j_{i-1} to $j_i = j_{i-1} + l_i$, then l_i is called the range of the i-th edge, and the r-tuploid

$$\big((l_i, \mu_1), \ldots, (l_r, \mu_r)\big)$$

is the Newton type of P with respect to u and a. The exact definition is this. If

$$j_0 := \min\{j \mid \ P_j \neq 0\}$$

and $j_1, \ldots, j_s$ and $\mu_1, \ldots, \mu_s$ are already defined, then either $j_s = n$, in which case $r := s$ finishes the construction, or

$$\mu_{s+1} := \min\Big\{ \frac{\nu_i - \nu_{j_s}}{i - j_s} \ \Big|\ i = j_s + 1, \ldots, n \Big\}$$

and

$$j_{s+1} = j_s + l_s := \max\Big\{ i \ \Big|\ \frac{\nu_i - \nu_{j_s}}{i - j_s} = \mu_{s+1} \Big\}.$$

2. Main results

The following theorems are in the line of the famous theorems of Malmquist ([Ma]) and Yosida ([Yo]) providing necessary conditions for the existence of an admissible solution of an algebraic differential equation. More precisely, they are generalizations of results due to K. Ishizaki [Is] concerning differential equations of the form

$$P_m(\alpha)\,(S\alpha)^m + P_0(\alpha) = 0,$$

where $m \in \mathbf{N}$, and P_m and P_0 are polynomials with meromorphic coefficients. Theorem 2.1 is exactly the same as in the cited article, except for the wider class of differential equations to which it applies. The idea of the proof is also due to K. Ishizaki; where he uses a result of A.Z. Mokhon'ko [Mo], we shall need the corresponding generalization by A.E. Eremenko [Er]. However, Theorem 2.3, which justifies the title of this article, will require considerably more work than its counterpart in [Is]. The idea of using the theory of valuations is also due to Eremenko ([Er]). A survey of the results of Malmquist, Yosida, and Ishizaki, and of many related topics is presented in [La].

Theorem 2.1. *Let α be an admissible solution of* (1.2)*, where $P \in \mathscr{A}_\varphi[s,t]$ is an irreducible polynomial of degree d in s and degree m in t. Then the Nevanlinna deficiency sum fulfills the inequality*

$$\sum_{c\in\widehat{\mathbf{C}}} \delta(c,\alpha) \le 2 - \frac{d}{2m}. \tag{2.1}$$

An immediate consequence is:

Corollary 2.2. *In the situation of Theorem* 2.1*, we must have $d \le 4m$.*

Notice that in the case of an algebraic differential equation having the ordinary derivative instead of the Schwarzian derivative, the corresponding condition is $d \le 2m$ (see [Er]).

Theorem 2.3. *Let*

$$P = P(s,t) = \sum_{j=0}^{m} P_j(s)\,t^j$$

be an irreducible polynomial over $\mathscr{A}_\varphi$, the leading coefficient of which has the factorization

$$P_m(s) = \prod_{i=1}^{\lambda}(s-c_i)\,\prod_{j=1}^{\Lambda}(s-\gamma_j),$$

where the c_i are constants and the γ_j are non-constant algebroid functions in $\mathscr{A}_\varphi$. For each $\gamma \in \{c_1,\dots,c_\lambda,\infty,\gamma_1,\dots,\gamma_\Lambda\}$, let P have a Newton type

$$\big((l_{\gamma,-R_\gamma},\mu_{\gamma,-R_\gamma}),\dots,(l_{\gamma,0},\mu_{\gamma,0}),(l_{\gamma,1},\mu_{\gamma,1}),\dots,(l_{\gamma,r_\gamma},\mu_{\gamma,r_\gamma})\big)$$

with respect to s *and* γ*. Let the indication be in such a way that* $\mu_{\gamma,1}$ *is the first positive slope, i. e.*

$$\mu_{\gamma,0} \le 0 < \mu_{\gamma,1}.$$

Then if (1.2) *has an admissible solution, the positive Newton slopes fulfill the following conditions*:

(a) *For* $c \in \{c_1, \ldots, c_\lambda\} \cup \{\infty\}$ *and* $q = 1, \ldots, r_c$: $\mu_{c,q} = 2/n_{c,q}$ *with* $n_{c,q} \in \{2, 3, \ldots\}$.

(b) *For* $\gamma = \{\gamma_1, \ldots, \gamma_\Lambda\}$: $r_\gamma = 1$ (*i. e., there is only one positive Newton slope*) *and* $\mu_{\gamma,1} = 2$.

(c) *The numbers* $n_{c,q}$ *in* (a) *satisfy*:

$$\sum_{c \in \{c_1, \ldots, c_\lambda, \infty\}} \sum_{q=1}^{r_c} l_{c,q} \Big(1 - \frac{1}{n_{c,q}}\Big) \le 2m - \sum_{\gamma \in \{\gamma_1, \ldots, \gamma_\Lambda\}} l_{\gamma,1}.$$

3. Lemmas

If $L = K(x,y)$ is an algebraic function field and $u \in L$ is transcendental over K, then the restriction of any valuation ν of L to $K(u)$ is a valuation of $K(u)$. There is a positive integer e, called the ramification index, such that

$$\nu(L^*) = \frac{1}{e}\nu\big(K(u)^*\big).$$

Conversely, the following lemma shows that each valuation of $K(u)$ has a continuation (as a valuation) on L. It can immediately be deduced from three theorems to be found in [Ca, VI §3, IX §2, VII §1].

Lemma 3.1. *Let* $w \in L$ *and* P *be the minimal polynomial of* (u, w)*, with Newton type*

$$\big((l_1, \mu_1), \ldots, (l_r, \mu_r)\big)$$

with respect to u *and* a*. Then the normalized continuations of* $\nu_{u,a}$ *onto* $K(u, w)$ *(possibly a smaller field than* L*) can be listed in the form*

$$\nu_i^j, \qquad i = 1, \ldots, r, \; j = 1, \ldots, s_i,$$

such that the following statements hold: *If* e_i^j *are the corresponding ramification indices, then:*

(i)
$$\frac{\nu_i^j(w)}{e_i^j} = -\mu_i.$$

(ii)
$$\sum_{j=0}^{s_i} e_i^j = l_i.$$

A divisor (of L over K) is a mapping $\delta\colon \mathcal{N}(L) \to \mathbf{Z}$ with $\delta(\nu) = 0$ for almost all ν. The degree of δ is

$$\operatorname{Deg}\delta = \sum_{\nu \in \mathcal{N}(L)} \delta(\nu).$$

For each $\nu \in \mathcal{N}(L)$, the characteristic divisor χ_ν is defined by

$$\chi_\nu(\nu) = 1, \qquad \chi_\nu(\sigma) = 0 \quad \text{for } \sigma \neq \nu.$$

For each $w \in L$, the associated divisor δ_w is defined by

$$\delta_w(\nu) = \nu(w).$$

As usual, the negative part of a divisor δ is

$$\delta^-(\nu) := -\min\{0, \delta(\nu)\}.$$

The following lemma is a conclusion from the Riemann–Roch theorem (see [Er]).

Lemma 3.2. *Let L be an algebraic function field over an algebraically closed field K of characteristic 0. Then there is a nonnegative integer $g = g(L/K)$ (called the genus of L) with the following property: For each integer $k \geq 2g$ and each normal valuation $\nu \in \mathcal{N}(L)$ there is an element $w \in L$ such that*

$$\delta_w^- = k\,\chi_\nu. \tag{3.1}$$

In the sequel, we fix $k := 2g + 1$ and call an element w satisfying (3.1) a k-element for ν.

Lemma 3.3. *Let w be a k-element for ν. Let u be another element of L and Q be the minimal polynomial of (w, u). Then the Newton type of Q with respect to w and ∞ has the form $((l, \mu))$ (i. e., the Newton polygon consists of one edge only), where*

$$\mu = -\frac{\nu(u)}{k}.$$

Proof. The only normal valuation of $K(w)$ to take a negative value at w is $\nu_{w,\infty}$. Thus the valuations of $\mathcal{N}(L)$ that take negative values at w are exactly the normalized continuations of $\nu_{w,\infty}$ onto L. By the assumption, ν is the only normal valuation with that property, so the only normalized continuation of $\nu_{w,\infty}$ onto L. This implies that there is only one such valuation of $K(w, u)$, too, which is just the normalized restriction of ν to $K(w, u)$. Let us call it ρ. By Lemma 3.1, the Newton type of Q can only be of the form $((l, \mu))$, where

$$\mu = \frac{\rho(u)}{-e} = \frac{\rho(u)}{\rho(w)}.$$

Since ρ and ν differ by a factor only,

$$\frac{\nu(u)}{-k} = \frac{\nu(u)}{\nu(w)} = \frac{\rho(u)}{\rho(w)} = \mu. \square$$

Remark 3.4. By the definition of $\nu_{w,\infty}$, the fact that

$$Q(w,u) = \sum_{j=0}^{n} Q_j(w)u^j$$

has the Newton type $((l,\mu))$ with respect to w and ∞ is equivalent with

$$\deg Q_j \le \deg Q_0 - j\,\mu, \; j = 0,\dots,n, \tag{3.2}$$
$$\deg Q_n = \deg Q_0 - n\,\mu. \tag{3.3}$$

In the setting of algebroid functions, this implies two useful lemmas on k-elements. For those, we assume $L \le \mathscr{A}$ to be an algebraic function field over $\mathscr{A}_\varphi$, $\nu \in \mathscr{N}(L)$, and β a k-element for ν.

Lemma 3.5. *Let $\zeta \in L$ be such that $\nu(\zeta) \ge 0$ and let $\pi\colon X \to \mathbf{C}$ be a covering such that there are on X simultaneous realizations of β, ζ, and each coefficient of the minimal polynomial Q of (β,ζ). Then there is a continuous function $F \in \mathscr{C}_X$ which satisfies:*

(i) $T(r,F) \approx O(\varphi(r))$.

(ii) *For each $x \in X$, it holds that $|\beta(x)| \le F(x)$ or $|\zeta(x)| \le F(x)$.*

Proof. By Lemma 3.3 and Remark 3.4, it holds that

$$\deg Q_j \le \deg Q_n,$$

since $\mu = -\nu(\zeta)/k \le 0$. Dividing by $Q_n(w)$, we get

$$\sum_{j=0}^{n} \frac{Q_j(\beta)}{Q_n(\beta)}\,\zeta^j = 0.$$

By elementary estimates (see [Er]), we obtain functions $F_j \in \mathscr{C}_X$, $j = 0,\dots,n$, such that

(i) $T(r,F_j) \approx O(\varphi(r))$.

(ii) For each $x \in X$, $|\beta(x)| \le F_j(x)$ or $\bigl|Q_j(\beta(x))/Q_n(\beta(x))\bigr| \le F_j(x)$.

Now let $F := n\max_j F_j$. Then $T(r,F) \approx O(\varphi(r))$ is obvious. For the second property, assume $|\beta(x)| \le F(x)$. Then

$$\left|\frac{Q_j(\beta(x))}{Q_n(\beta(x))}\right| \le \frac{1}{n}F(x), \qquad j = 0,\dots,n.$$

In particular, $F(x) \geq n$. Since $\zeta(x)$ is a zero of the polynomial

$$\sum_{j=0}^{n} \frac{Q_j\big(\beta(x)\big)}{Q_n\big(\beta(x)\big)} \zeta^j,$$

we must have $|\zeta(x)| \leq F(x)$. □

Let us fix some terminology. First, if n is a positive integer, we mean by a pole of order $-n$ a zero of order n. By a pole of order 0, we mean a point which is neither a pole nor a zero. Second, let Q be a polynomial in two unknowns over $\mathscr{A}_\varphi$, and π: $X \to \mathbf{C}$ a covering such that there are on X realizations of all the coefficients of Q. We will call a point $x \in X$ special with respect to Q (and the realizations of the coefficients) if it is a ramification point of π (i. e. a multiple point) or if it is a zero or a pole of one of the realizations of the coefficients.

Lemma 3.6. *Let η be an arbitrary element of L and Q the minimal polynomial of (β, η). Let π: $X \to \mathbf{C}$ be a covering such that there are on X realizations of β, η, and all the coefficients of Q, and let $x \in X$ be non-special. Assume further x to be a pole of order $\omega > 0$ of β_X. Then x is a pole of order*

$$-\frac{\nu(\eta)\,\omega}{k}$$

of η_X.

Proof. By Lemma 3.3 and Remark 3.4, we have

$$\deg Q_j \leq \deg Q_0 - j\,\mu,\ \ j = 0, \dots, n,$$
$$\deg Q_n = \deg Q_0 - n\,\mu,$$

where

$$\mu = -\frac{\nu(\eta)}{k}.$$

As the sum

$$\sum_{j=0}^{n} Q_j(\beta)\,\eta^j$$

of meromorphic functions on X vanishes identically, it is trivial that the maximal pole order in x of the summands $Q_j(\beta)\,\eta^j$ must occur at least twice. Since x is non-special, this is only possible if x is a pole of η_X of order $\mu\omega$. □

Next, we cite the result of Eremenko mentioned above.

Theorem 3.7 (Eremenko). *Let α and β be algebroid functions, and*

$$\varphi(r) \approx o\big(T(r, \alpha)\big).$$

Assume that there is an irreducible polynomial P in two unknowns over $\mathscr{A}_\varphi$ such that $P(\alpha, \beta) = 0$. Then $L := \mathscr{A}_\varphi(\alpha, \beta)$ is an algebraic function field over $\mathscr{A}_\varphi$, and the following relation holds:

$$\big(\mathrm{Deg}\ \delta_\beta^- + o(1)\big)\, T(r, \alpha) \approx (\mathrm{Deg}\ \delta_\alpha^-)\, T(r, \beta).$$

The next lemma is a consequence of Lemma 3.1.

Lemma 3.8. *In the situation of Theorem* 3.7, *we have*

$$\operatorname{Deg}\,\delta_\alpha^- = \deg_\beta P \qquad \text{and} \qquad \operatorname{Deg}\,\delta_\beta^- = \deg_\alpha P.$$

So we get the following generalization of results by Valiron [Va] and A.Z. Mokhon'ko [Mo].

Theorem 3.9. *Let* α *and* β *be algebroid functions, and*

$$\varphi(r) \approx o\bigl(T(r,\alpha)\bigr).$$

Let P *be an irreducible polynomial in two unknowns over* $\mathscr{A}_\varphi$ *such that* $P(\alpha,\beta) = 0$. *Then*

$$\bigl(\deg_\alpha P + o(1)\bigr)T(r,\alpha) \approx (\deg_\beta P)\,T(r,\beta).$$

We finish this section by listing some properties of the Schwarzian derivative. Let α be a non-constant algebroid function.

(a) If l is a Möbius transformation and $\tilde\alpha = l \circ \alpha$, then $S\tilde\alpha = S\alpha$.

(b) Let $\pi\colon X \to \mathbf{C}$ be a covering such that there is a realization of α, and therefore also of $S\alpha$, on X. If x is a ramification point of π, the pole order of $S\alpha$ at x is at most $2\operatorname{ram}(x,\pi)$. If y is not a ramification point, it is a pole if and only if the order of α at x is greater than 1. In this case, the pole order is always 2.

(c) The lemma on the logarithmic derivative immediately yields $m(r,S\alpha) \approx o\bigl(T(r,\alpha)\bigr)$.

(d) Items (b) and (c) imply

$$T(r,S\alpha) \approx 2\,\overline{N}_1(r,\alpha) + o\bigl(T(r,\alpha)\bigr) + O\bigl(N_{\mathrm{ram}}(r,\pi)\bigr), \tag{3.4}$$

where α is identified with its realization on X.

4. Proof of the results

Let α be an admissible solution of (1.2). Then $L = \mathscr{A}_\varphi(\alpha, S\alpha)$ is an algebraic function field over $\mathscr{A}_\varphi$. By the definition of an admissible solution respectively by the ramification theorem of algebroid functions (see [Ul]), the counting functions of the ramification points of α and of any element of $\mathscr{A}_\varphi$ are of growth $\approx O\bigl(\varphi(r)\bigr)$. It is easy to see that the same applies for every element of L. For the proofs, we assume that $\pi\colon X \to \mathbf{C}$ is a covering such that there are on X realizations of all algebroid functions involved. These are finitely many elements of L. We can choose π in such a way that it is also an element of L, in particular

$$N_{\mathrm{ram}}(r,\pi) \approx O\bigl(\varphi(r)\bigr).$$

Proof of Theorem 2.1. By the assumption and Theorem 3.9, we obtain

$$T(r,S\alpha) \approx \Bigl(\frac{d}{m} + o(1)\Bigr)\,T(r,\alpha).$$

With (3.4) and property (1.4) of an admissible solution, this yields

$$2\,\overline{N_1}(r,\alpha) \approx \Big(\frac{d}{m} + o(1)\Big) T(r,\alpha).$$

As an immediate consequence of the second fundamental theorem, we obtain for q distinct values $c_1, \ldots, c_q \in \widehat{\mathbf{C}}$:

$$\begin{aligned}\sum_{i=1}^{q} \delta(c_i,\alpha) &= \sum_{i=0}^{q} \liminf_{r\to\infty} \frac{m(r,c_i,\alpha)}{T(r,\alpha)} \le \liminf_{r\to\infty} \frac{2\,T(r,\alpha) - N_1(r,\alpha)}{T(r,\alpha)} \\ &\le \liminf_{r\to\infty} \frac{2\,T(r,\alpha) - \overline{N}_1(r,\alpha)}{T(r,\alpha)} \le 2 - \frac{d}{2m}.\end{aligned}$$

This implies (2.1). □

Proof of Theorem 2.3. Let us first get rid of ν_∞. The Möbius transformation

$$\tilde{\alpha} := l \circ \alpha,$$

yields a differential equation

$$\widetilde{P}(\tilde{\alpha}, S\tilde{\alpha}) = 0$$

where the Newton type of P with respect to α and γ equals the Newton type of $\widetilde{P}$ with respect to $\tilde{\alpha}$ and $l(\gamma)$. By choosing l appropriately, we get a Newton polygon with respect to ν_∞ consisting of one edge of slope 0. So we may as well assume that we have this situation from the beginning. Let $\gamma \in \{c_1, \ldots, c_\lambda, \gamma_1, \ldots, \gamma_\Lambda\}$ and μ be a positive slope of P with respect to α and γ with range l of the corresponding edge. Lemma 3.1 yields: There are normalized continuations $\nu^1, \ldots, \nu^s$ of $\nu_{\alpha,\gamma}$ onto L with ramification numbers $e^1, \ldots, e^s$ such that

$$\frac{\nu^j(S\alpha)}{e^j} = -\mu$$

and

$$\sum_{j=0}^{s} e^j = l. \tag{4.1}$$

Let $\nu \in \{\nu^1, \ldots, \nu^s\}$ and $e = \nu(\alpha - \gamma)$ be the ramification number of ν. There is a k-element $\beta = \beta_\nu$ for ν. We will collect some properties of β.

1. By Lemma 3.8, Deg $\delta_\alpha^- = \deg_{S\alpha} P = m$. By the choice of β, Deg $\delta_\beta^- = k$. So Theorem 3.7 yields

$$T(r,\beta) \approx \Big(\frac{k}{m} + o(1)\Big)\, T(r,\alpha). \tag{4.2}$$

2. Since $\nu(S\alpha) < 0$, there is a positive integer n such that $\nu\big(\beta/(S\alpha)^n\big) \geq 0$. By Lemma 3.5, there is $f \in \mathscr{C}_X$ such that
(i) $T(r, f) \approx O\big(\varphi(r)\big)$.
(ii) For each $x \in X$, $|\beta(x)| \leq f(x)$ or $|(\beta/(S\alpha)^n)(x)| \leq f(x)$.
This implies

$$m(r, \beta) \approx O\big(\varphi(r)\big) + m\big(r, (S\alpha)^n\big) \approx o\big(T(r, \alpha)\big).$$

In combination with (4.2), this yields

$$N(r, \beta) \approx \Big(\frac{k}{m} + o(1)\Big)\, T(r, \alpha). \tag{4.3}$$

3. In the sequel, special and non-special will always be used with respect to the minimal polynomials of (β, α) and $(\beta, S\alpha)$ simultaneously. We denote by $N_{\mathrm{spe}}(r, \beta)$ and $N_{\mathrm{non}}(r, \beta)$ the counting functions of the poles of β that are special respectively non-special. Let the minimal polynomial of $(\beta, S\alpha)$ be

$$Q(\beta, S\alpha) = \sum_{j=0}^{p} Q_j(\beta)\, (S\alpha)^j.$$

By Lemma 3.3, the Newton polygon of Q with respect to β and ∞ consists of one edge of positive slope. By Remark 3.4,

$$\deg Q_0 > \deg Q_j \qquad \text{for } j = 1, \ldots, p.$$

So if we sort Q in powers of β, the coefficient of the highest power is a polynomial in $S\alpha$ of degree 0, which we can assume to be 1. It is elementary that the order of a pole of β at a point x is bounded by the maximal pole order of the coefficients ζ_i of Q (elements of $\mathscr{A}_\varphi$) plus p times the pole order of $S\alpha$. But the pole order of $S\alpha$ is at most $2q$, q being the number of sheets of π. This yields

$$N_{\mathrm{spe}}(r, \beta) \leq 2qp\overline{N}_{\mathrm{spe}}(r, \beta) + \sum_i N(r, \zeta_i) \approx O\big(\varphi(r)\big),$$

where the bar indicates counting ignoring multiplicities. So with (4.3), we obtain

$$N_{\mathrm{non}}(r, \beta) \approx \Big(\frac{k}{m} + o(1)\Big)\, T(r, \alpha).$$

4. By Lemma 3.6, each non-special pole of β of order ω is a zero of $\alpha - \gamma$ of order

$$n = \frac{e\,\omega}{k}$$

and a pole of $S\alpha$ of order μn. So we must have $\mu n = 2$, which yields

$$\mu = \frac{2}{n}. \tag{4.4}$$

Notice that n is the same for all non-special poles, so

$$\overline{N}_{\mathrm{non}}(r,\beta) \approx \Big(\frac{e}{mn} + o(1)\Big)\, T(r,\alpha). \tag{4.5}$$

5. Let $\tilde{\beta}$ be a k-element for $\tilde{\nu} \neq \nu$. We denote by $N_{\tilde{\beta}}(r,\beta)$ the counting function of poles of β that are also poles of $\tilde{\beta}$. By Lemma 3.5, there is $f \in \mathscr{C}_X$ fulfilling

(i) $T(r,f) \approx O\big(\varphi(r)\big)$.
(ii) For each $x \in X$, $|\beta(x)| \le f(x)$ or $|\tilde{\beta}(x)| \le f(x)$.
By item 4 of this proof, the order of non-special poles of β is bounded, so this yields

$$N_{\tilde{\beta}}(r,\beta) \approx O\big(\varphi(r)\big).$$

We now proceed to prove (a), (b), and (c). Item 4 of this proof covers half of (a) and (b). For the rest, we only have to show that $n \ge 2$ if γ is constant and $n = 1$ if γ is non-constant, n being the order of the zero of $\alpha - \gamma$ at a non-special pole of β.

In the constant case, if x is a non-special pole of β, then n is equal to the multiplicity of α at x. Since $S\alpha(x) = \infty$, we must have $n \ge 2$.

If γ is non-constant, we have to show that there is a non-special pole of β that is a simple zero of $\alpha - \gamma$. By (4.5) and $T(r,\gamma) \approx O\big(\varphi(r)\big)$, there is certainly a non-special pole x where $\gamma(x) \neq \infty$ and $\gamma'(x) \neq 0$. By item 4, x is a zero of $\alpha - \gamma$ and a pole of $S\alpha$, so $\alpha(x) = \gamma(x) \neq \infty$ and $\alpha'(x) = 0$. This yields multiplicity $n = 1$.

For (c), we count the zeros of α' and make use of the fact that

$$N\Big(r, \frac{1}{\alpha'}\Big) \lesssim \big(2 + o(1)\big)\, T(r,\alpha). \tag{4.6}$$

Denote by $N_\beta(r, 1/\alpha')$ the counting function of zeros of α' that coincide with non-special poles of β. If β belongs to a constant γ,

$$N_\beta\Big(r,\frac{1}{\alpha'}\Big) \approx (n-1)\overline{N}_{\mathrm{non}}(r,\beta) \approx (n-1)\Big(\frac{e}{mn} + o(1)\Big)\, T(r,\alpha), \tag{4.7}$$

where n depends on β. If β belongs to a non-constant γ, we know by $T(r,\gamma) \approx O\big(\varphi(r)\big)$ that non-special poles are zeros of α' except for some in the order of $O\big(\varphi(r)\big)$ (see the proof of (b) above). Thus

$$N_\beta\Big(r,\frac{1}{\alpha'}\Big) \gtrsim \overline{N}_{\mathrm{non}}(r,\beta) \approx \Big(\frac{e}{m} + o(1)\Big)\, T(r,\alpha). \tag{4.8}$$

By item 5, k-elements for different ν have few non-special poles in common, so we may add the $N_\beta(r,1/\alpha')$ to get a lower estimate of $N(r,1/\alpha')$,

$$\sum_\beta N_\beta\Big(r,\frac{1}{\alpha'}\Big) + O\big(\varphi(r)\big) \lesssim N\Big(r,\frac{1}{\alpha'}\Big),$$

where the sum ranges over the k-elements for all continuations of any positive Newton slope of any c_i or γ_j. With (4.7), (4.8), and (4.1), this yields

$$\frac{1}{m}\left[\left(\sum_{i=1}^{\lambda}\sum_{q=1}^{r_i} l_{i,q}\left(1-\frac{1}{n_{i,q}}\right)\right)+\sum_{i=1}^{\Lambda} L_{i,1}+o(1)\right] T(r,\alpha) \lesssim N\left(r,\frac{1}{\alpha'}\right).$$

In combination with (4.6), this yields (c). □

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