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AN INHOMOGENEOUS DIRICHLET PROBLEM FOR A NON-HYPOELLIPTIC LINEAR PARTIAL DIFFERENTIAL OPERATOR

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Abstract. In this paper we state an inhomogeneous Dirichlet problem for a class of linear partial differential operators which are non-hypoelliptic. We prove uniqueness, existence and regularity results for its solutions.

1. Introduction

In [1] K. Doppel and the present author stated a homogeneous Dirichlet problem for non-hypoelliptic linear partial differential operators, especially for a product of uniformly elliptic differential operators with smooth coefficient functions. Here we state an inhomogeneous Dirichlet problem for the same class of partial differential operators. The related homogeneous problem turns out to be an equivalent reformulation of the homogeneous Dirichlet problem in [1]. But this new formulation yields more general results than the former one. For further details of this problem and for some literature relevant in this context we refer to the introduction in [1].

2. The inhomogeneous problem

Most of the definitions used here are taken from [1]. We recapitulate only some notation necessary for understanding the subsequent theorems.

Let Ω_1 and Ω_2 be bounded domains in $\mathbf{R}^{n_1}$ or $\mathbf{R}^{n_2}$ ($n_1, n_2 \geq 2$, respectively) with boundaries $\partial\Omega_\mu$ ($\mu = 1, 2$) of class C^∞ (cf. e.g. Grisvard [2, Definition 1.2.1.1]). Thus the domains Ω_μ satisfy the uniform cone condition and the product domain $\Omega := \Omega_1 \times \Omega_2 \subset \mathbf{R}^n$ ($n = n_1 + n_2$) has the same property (cf. Hochmuth [3, Satz 3.1]). Furthermore, Ω is a Lipschitz domain (cf. Grisvard [2, Theorem 1.2.2.2]).

On each of the domains Ω_μ we consider a uniformly elliptic differential operator $P_\mu(\,\cdot\,, D_{x_\mu})$ of second order

$$P_\mu(\,\cdot\,, D_{x_\mu}) := -\sum_{i,j=1}^{n_\mu} D_j\big(a_{ij}^{(\mu)}(\cdot) D_i\big) + \sum_{i=1}^{n_\mu} b_i^{(\mu)}(\cdot) D_i + c^{(\mu)}(\cdot)$$

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with $D_{x_\mu} = (D_1, \ldots, D_{n_\mu})$ and $D_j = \partial/\partial x_j$, where $a_{ij}^{(\mu)}, b_i^{(\mu)}, c^{(\mu)} \in C^\infty(\overline{\Omega}_\mu)$ are given real-valued functions with $a_{ij}^{(\mu)} = a_{ji}^{(\mu)}$. Note that then there are constants $\varrho_\mu \in \mathbf{R}^+$ with

$$\sum_{i,j=1}^{n_\mu} a_{ij}^{(\mu)}(x_\mu)\xi_i\xi_j \geq \varrho_\mu \sum_{i=1}^{n_\mu} \xi_i^2 \qquad \text{for all } x_\mu \in \Omega_\mu \text{ and } (\xi_1, \ldots, \xi_{n_\mu}) \in \mathbf{R}^{n_\mu}.$$

By $\widetilde{P}_\mu(\,\cdot\,, D_{x_\mu})$ we denote the formal adjoint operators

$$\widetilde{P}_\mu(\,\cdot\,, D_{x_\mu}) := -\sum_{i,j=1}^{n_\mu} D_j\big(a_{ji}^{(\mu)}(\cdot)D_i\big) - \sum_{i=1}^{n_\mu} D_i\big(b_i^{(\mu)}(\cdot)\big) + c^{(\mu)}(\cdot).$$

We consider the classical homogeneous elliptic Dirichlet problems:

Problem (Ω_μ) $(\mu = 1, 2)$. *For $f_\mu \in C^{0;\lambda}(\Omega_\mu) \cap C(\overline{\Omega}_\mu)$ $(\lambda \in (0,1])$ find a function $u_\mu \in C^2(\Omega_\mu) \cap C(\overline{\Omega}_\mu)$ such that*

$$\begin{aligned} P_\mu(x_\mu, D_{x_\mu})u_\mu(x_\mu) &= f_\mu(x_\mu) && \text{for } x_\mu \in \Omega_\mu, \\ u_\mu(x_\mu) &= 0 && \text{for } x_\mu \in \partial\Omega_\mu. \end{aligned}$$

Now we formulate the inhomogeneous Dirichlet problem of classical type for the non-hypoelliptic product operator $P(\,\cdot\,, D_x)$ defined by

$$P(x, D_x) := P_1(x_1, D_{x_1})P_2(x_2, D_{x_2}) \qquad \text{for } x = (x_1, x_2) \in \Omega_1 \times \Omega_2.$$

For $P(\,\cdot\,, D_x)$ the formal adjoint operator is denoted by $\widetilde{P}(\,\cdot\,, D_x)$.

Problem (Ω). *For $f \in C(\Omega)$ and $g \in C(\partial\Omega)$ find a function $u \in C^{2,2}(\Omega) \cap C(\overline{\Omega})$ such that*

$$\begin{aligned} P(x, D_x)u(x) &= f(x) && \text{for } x \in \Omega, \\ u(x) &= g(x) && \text{for } x \in \partial\Omega \end{aligned}$$

is valid.

To give a Hilbert space formulation of problem (Ω) we define

$$\Gamma := \{\sigma_1 \in \mathbf{N}_0^{n_1} \mid |\sigma_1| \leq 1\} \times \{\sigma_2 \in \mathbf{N}_0^{n_2} \mid |\sigma_2| \leq 1\}. \tag{2.1}$$

A simple calculation shows that there are functions $a_{\sigma\tau} \in C^\infty(\overline{\Omega})$ $(\sigma, \tau \in \Gamma)$ with which we can write the operator $P(\,\cdot\,, D_x)$ in the form

$$P(\,\cdot\,, D_x) = \sum_{\sigma,\tau \in \Gamma} (-1)^{|\sigma|} D^\sigma\big(a_{\sigma\tau}(\cdot)D^\tau\big). \tag{2.2}$$

Thus, partial integration on $C^\infty(\overline{\Omega}) \times C_0^\infty(\Omega)$ gives the bilinear form $b(\,\cdot\,,\cdot\,)$:

$$b(u,\phi) := \big(P(\,\cdot\,,D_x)u,\phi\big)_{0;\Omega} = \sum_{\sigma,\tau\in\Gamma} (a_{\sigma\tau}D^\tau u, D^\sigma \phi)_{0;\Omega}. \tag{2.3}$$

We denote by $\alpha^\mu = (\alpha_1^\mu,\dots,\alpha_n^\mu) \in \mathbf{N}_0^n$ ($\mu = 1,2$) multi-indices for which $\alpha_j^1 = 0$ for all $j \in \{n_1+1,\dots,n\}$ and $\alpha_k^2 = 0$ for all $k \in \{1,\dots,n_1\}$, respectively, and define the (anisotropic) Sobolev space $H^{s,t}(\Omega)$ ($s,t \in \mathbf{N}_0$) as the completion of $C_*^{s+t}(\Omega)$ with respect to the norm

$$\|u\|_{s,t;\Omega} := \Bigg(\sum_{\substack{\alpha=\alpha^1+\alpha^2\\ |\alpha^1|\le s,\,|\alpha^2|\le t}} \|D^\alpha u\|_{0;\Omega}^2\Bigg)^{1/2},$$

and $H_0^{1,1}(\Omega)$ as the smallest closed subspace of $H^{1,1}(\Omega)$ including $C_0^\infty(\Omega)$ (cf. [1]).

Because of the boundedness of the functions $a_{\sigma\tau}$, there exists a constant $c \in \mathbf{R}^+$ with which

$$|b(u,\varphi)| \le c\|u\|_{1,1;\Omega}\|\varphi\|_{1,1;\Omega} \qquad \text{for all } (u,\varphi) \in C^\infty(\overline{\Omega}) \times C_0^\infty(\Omega),$$

and the bilinear form $b(\,\cdot\,,\cdot\,)$ can be continuously extended to $H^{1,1}(\Omega) \times H_0^{1,1}(\Omega)$. This extension will also be denoted by $b(\,\cdot\,,\cdot\,)$. Because the product domain Ω has a Lipschitz boundary $\partial\Omega$ and because of the continuous embedding $H^{1,1}(\Omega) \hookrightarrow H^1(\Omega)$ we can introduce the trace operator $\gamma\colon H^{1,1}(\Omega) \to H^{1/2}(\partial\Omega)$ (cf. Grisvard [2, Theorem 1.5.1.3]).

For the weak formulation of the inhomogeneous Dirichlet problem (Ω) we call an *admissible boundary function* every $g \in L^2(\Omega)$ for which the through

$$\ell_g(\varphi) := \big(g, \widetilde{P}(\,\cdot\,,D_x)\varphi\big)_{0;\Omega} \qquad \text{for all } \varphi \in C_0^\infty(\Omega)$$

dense in $H_0^{1,1}(\Omega)$ defined linear functional is bounded with respect to $\|\cdot\|_{1,1;\Omega}$, i.e. with some constant $c \in \mathbf{R}^+$ one has

$$\big|\big(g, \widetilde{P}(\,\cdot\,,D_x)\varphi\big)_{0;\Omega}\big| < c\|\varphi\|_{1,1;\Omega} \qquad \text{for all } \varphi \in C_0^\infty(\Omega).$$

Problem (W). *For $f \in L_2(\Omega)$ and for an admissible boundary function $g \in L_2(\Omega)$ find a function $u \in L_2(\Omega)$ such that $w := u - g \in H^{1,1}(\Omega)$ fulfills the equations*

$$b(w,\varphi) = (f,\varphi)_{0;\Omega} + \big(g, \widetilde{P}(\,\cdot\,,D_x)\varphi\big)_{0;\Omega} \qquad \textit{for all } \varphi \in C_0^\infty(\Omega),$$
$$\gamma w = 0.$$

For the present it seems that the related homogeneous problem with $g = 0$ differs from the weak problem (B) in [1]. However, the two formulations are equivalent (cf. Theorem 3.2). But, by the one given here, one can easily discern that the star-shapedness of the domain Ω assumed in [1] is not necessary for the fact that the solutions of problem (Ω) are also solutions of problem (W). Furthermore, it is easier to handle the inhomogeneous problem by this formulation.

3. Uniqueness and existence of solutions

First we state a uniqueness result.

Theorem 3.1. *Assume that problems* (Ω_μ) $(\mu = 1, 2)$ *are uniquely solvable. Then a function* $w \in H^{1,1}(\Omega)$ *with* $\gamma w = 0$ *and*

$$b(w, \varphi) = 0 \qquad \text{for all } \varphi \in C_0^\infty(\Omega)$$

vanishes, i.e. $w = 0$.

Proof. Let $\varphi_\mu \in C_0^\infty(\Omega_\mu)$ $(\mu = 1, 2)$ and $\varphi := \varphi_1 \otimes \varphi_2 \in C_0^\infty(\Omega)$. Then partial integration and the theorem of Fubini give

$$\begin{aligned} 0 &= b(w, \varphi) = (w, \widetilde{P}_2 \widetilde{P}_1 \varphi)_{0;\Omega} \\ &= \int_{\Omega_2} \widetilde{P}_2 \varphi_2(x_2) \Big(\int_{\Omega_1} w(x_1, x_2) \widetilde{P}_1 \varphi_1(x_1)\, dx_1 \Big)\, dx_2 = (\widetilde{w}, \widetilde{P}_2 \varphi_2)_{0;\Omega_2} \end{aligned} \tag{3.1}$$

with $\widetilde{w} \in H^1(\Omega_2)$ defined by $\widetilde{w}(x_2) := \big(w(\,\cdot\,, x_2), \widetilde{P}_1 \varphi_1\big)_{0;\Omega_1}$ for $x_2 \in \Omega_2$.

Because of $\gamma w = 0$, i.e. especially

$$\int_{\partial\Omega_2} \Big(\int_{\Omega_1} |w(x_1, \eta)|^2\, dx_1 \Big)\, d\eta = 0,$$

we get for almost all $\eta \in \partial\Omega_2$

$$\int_{\Omega_1} |w(x_1, \eta)|^2\, dx_1 = 0. \tag{3.2}$$

Bearing in mind (3.2) and observing

$$|\widetilde{w}(x_2)| = \big|\big(w(\,\cdot\,, x_2), \widetilde{P}_1 \varphi_1\big)_{0;\Omega_1}\big| \le \|w(\,\cdot\,, x_2)\|_{0;\Omega_1} \|\widetilde{P}_1 \varphi_1\|_{0;\Omega_2}$$

we get $\widetilde{w} \in H_0^1(\Omega_2)$. Therefore relation (3.1), the unique solvability of problem (Ω_2) and elliptic regularity yield $\widetilde{w} = 0$, and we have for $x_2 \in \Omega_2$

$$\big(w(\,\cdot\,, x_2), \widetilde{P}_1 \varphi_1\big)_{0;\Omega_1} = 0 \qquad \text{for } \varphi_1 \in C_0^\infty(\Omega_1).$$

Furthermore, we have $w(\,\cdot\,, x_2) \in H_0^1(\Omega_1)$ for almost all $x_2 \in \Omega_2$. Finally, the unique solvability of problem (Ω_1) and the elliptic regularity give $w(\,\cdot\,, x_2) = 0$ for almost all $x_2 \in \Omega_2$, i.e. $w = 0$. □

The next theorem shows that $H_0^{1,1}(\Omega)$ can be characterized by the trace operator γ. A consequence of this theorem is that problem (B) in [1] and problem (W) with $g = 0$ are equivalent.

Theorem 3.2. *A function $w \in H^{1,1}(\Omega)$ lies in $H_0^{1,1}(\Omega)$ if and only if $\gamma w = 0$.*

Proof. Because $C_0^\infty(\Omega)$ is dense in $H_0^{1,1}(\Omega)$, the functions $w \in H_0^{1,1}(\Omega)$ have the property $\gamma w = 0$. Thus we only have to prove that functions $w \in H^{1,1}(\Omega)$ with $\gamma w = 0$ lie in $H_0^{1,1}(\Omega)$. First we remark that the bilinear form $b(\,\cdot\,,\cdot) := (\,\cdot\,,\cdot)_{1,1;\Omega}$ is induced by the Laplace operators $P_\mu(\,\cdot\,, D_{x_\mu}) = -\Delta$. The boundary value problems (Ω_μ) with respect to this partial differential operators are uniquely solvable. On the other hand, for $w \in H^{1,1}(\Omega)$ a unique $\widetilde{w} \in H_0^{1,1}(\Omega)$ exists with

$$(w - \widetilde{w}, \varphi)_{1,1;\Omega} = 0 \qquad \text{for all } \varphi \in H_0^{1,1}(\Omega).$$

By $\gamma(w - \widetilde{w}) = 0$ and Theorem 3.1 we have $w = \widetilde{w}$, i.e. $w \in H_0^{1,1}(\Omega)$. □

To state an existence result we have to introduce the coerciveness: we say that the bilinear form $b(\,\cdot\,,\cdot)$ is $H_0^{1,1}(\Omega)$-coercive if there are constants $\varrho \in \mathbf{R}^+$ and $q \in \mathbf{R}$ with which

$$b(u,u) \geq \varrho \|u\|_{1,1;\Omega}^2 - q\|u\|_{0;\Omega}^2 \qquad \text{for all } u \in H_0^{1,1}(\Omega).$$

Theorem 3.3. *If problems (Ω_μ) are uniquely solvable and if the bilinear form $b(\,\cdot\,,\cdot)$ is $H_0^{1,1}(\Omega)$-coercive, then for every $f \in L_2(\Omega)$ and for every admissible boundary function $g \in L_2(\Omega)$ there exists a unique solution $u = w + g \in L_2(\Omega)$ of problem* (W) *with $w \in H_0^{1,1}(\Omega)$.*

Proof. The inhomogeneous problem is equivalent to finding a function $w \in H_0^{1,1}(\Omega)$ (cf. Theorem 3.2) with

$$b(w,\varphi) = (f,\varphi)_{0;\Omega} + \big(g, \widetilde{P}(\,\cdot\,, D_x)\varphi\big)_{0;\Omega} \qquad \text{for all } \varphi \in H_0^{1,1}(\Omega).$$

The $H_0^{1,1}(\Omega)$-coerciveness of the bilinear form $b(\,\cdot\,,\cdot)$ implies the Fredholm property for the latter problem. Therefore, because of Theorem 3.1, there is a unique solution $w \in H_0^{1,1}(\Omega)$. Thus the assertion of the theorem follows with $u = w + g$. □

Remark. Especially every function $g \in H^{1,1}(\Omega)$ is an admissible boundary function. For $g \in H^{1,1}(\Omega)$ the proof of Theorem 3.3 implies the existence of a constant $c \in \mathbf{R}^+$ independent of f and g with which

$$\|w\|_{1,1;\Omega} \leq c\big(\|f\|_{0;\Omega} + \|g\|_{1,1;\Omega}\big). \tag{3.3}$$

Furthermore, one gets $u \in H^{1,1}(\Omega)$ and the estimate (3.3) for solutions u.

4. Regularity of solutions

Obviously, the solutions $u \in C^\infty(\overline{\Omega})$ of the homogeneous boundary value problem considered in [1, Theorem 3.5], are also solutions of problem (W) for $g = 0$. Analogously to [1] one can show by (3.3) and the density of $C^\infty(\overline{\Omega})$ in $H^{s,t}(\Omega)$ (cf. Grisvard [2, Theorem 1.4.2.1]):

Theorem 4.1. *Let us suppose that problems* (Ω_μ) *are uniquely solvable and that the bilinear form* $b(\cdot,\cdot)$ *is* $H_0^{1,1}(\Omega)$*-coercive. Let further* $f \in L_2(\Omega)$ *and an admissible boundary function* $g \in L_2(\Omega)$ *be such that for some* $s, t \in \mathbf{N}_0$ *there is a constant* $c \in \mathbf{R}^+$ *with*

$$\left|(f,\varphi)_{0;\Omega} + \left(g, \widetilde{P}(\cdot, D_x)\varphi\right)_{0;\Omega}\right| \le c\|\varphi\|_{s,t;\Omega} \qquad \text{for all } \varphi \in C^\infty(\overline{\Omega}). \tag{4.1}$$

Then there exists a unique solution $u = w + g \in L_2(\Omega)$ *of problem* (W) *with*

$$w \in H^{s+2,t+2}(\Omega).$$

Remark. Condition (4.1) is obviously satisfied for functions $f \in H^{s,t}(\Omega)$ and $g \in H^{s+2,t+2}(\Omega)$. In this case one gets $u \in H^{s+2,t+2}(\Omega)$ and a constant $c_{s,t} \in \mathbf{R}^+$ independent of f and g with

$$\|u\|_{s+2,t+2;\Omega} \le c_{s,t}\left(\|f\|_{s,t;\Omega} + \|g\|_{s+2,t+2;\Omega}\right).$$

By using Theorem 4.1, the remark above, the continuous embedding $H^{s,t}(\Omega) \hookrightarrow H^{\min(s,t)}(\Omega)$ and the usual Sobolev embedding one gets regularity results similar to the classical ones. But the product property of the domain Ω allows us to prove more appropriate embedding and regularity theorems.

Theorem 4.2. *For the domains* $\Omega_\mu \subset \mathbf{R}^{n_\mu}$ $(\mu = 1, 2)$ *satisfying the uniform cone condition and for* $\Omega = \Omega_1 \times \Omega_2$ *we have a continuous embedding*

$$H^{s_1,s_2}(\Omega) \hookrightarrow C(\overline{\Omega})$$

for numbers $s_\mu \in \mathbf{N}_0$ *with* $s_\mu > n_\mu/2$.

Proof. By assumption also the domain Ω satisfies the uniform cone condition and therefore $C^\infty(\overline{\Omega})$ lies dense in $H^{s_1,s_2}(\Omega)$. Thus it is enough to prove for the functions $u \in C^\infty(\overline{\Omega})$ the estimate

$$\sup_{x \in \overline{\Omega}} |u(x)| \le c\|u\|_{s_1,s_2;\Omega} \tag{4.2}$$

for some constant $c \in \mathbf{R}^+$. Then continuation yields (4.2) on $H^{s_1,s_2}(\Omega)$, and thus the statement of the theorem is valid.

To give a detailed proof we have to introduce some notation concerning the assumed uniform cone condition. Here cones $C(r_\mu, \Sigma_\mu)$ with vertex at $x_\mu = 0$ are sets

$$C(r_\mu, \Sigma_\mu) := B(r_\mu) \cap \{\lambda y_\mu \mid y_\mu \in \Sigma_\mu, \lambda > 0\},$$

where Σ_μ is an open set on the surface of the ball $B(r_\mu) = \{y_\mu \in \mathbf{R}^{n_\mu} \mid |y_\mu| < r_\mu\}$.

Let $x = (x_1, x_2) \in \overline{\Omega}_1 \times \overline{\Omega}_2 \subset \overline{\Omega}$ and let $C_{x_\mu} = C(r_\mu, \Sigma_\mu)$ be cones related to the uniform cone condition with respect to Ω_μ, especially $x_\mu - C_{x_\mu} \subset \Omega_\mu$.

Furthermore, let f_μ be fixed functions in $C_0^\infty(\mathbf{R}^{n_\mu})$ with the properties

1. $f_\mu(y_\mu) = 1$ for $|y_\mu| < 1$,
2. $f_\mu(y_\mu) = 0$ for $|y_\mu| \geq 2$,
3. $|D^\alpha f_\mu(y_\mu)| \leq M$ for $y_\mu \in \mathbf{R}^{n_\mu}$, $|\alpha| \leq s_\mu + 1$ and with some $M \in \mathbf{R}^+$.

We define the functions $e_\mu(y_\mu) := f_\mu\big(2(y_\mu - x_\mu)/r_\mu\big)$ on $\mathbf{R}^{n_\mu}$. Thus we have

$$e_\mu(y_\mu) = \begin{cases} 1 & \text{for } |y_\mu - x_\mu| < r_\mu/2, \\ 0 & \text{for } |y_\mu - x_\mu| \geq r_\mu \end{cases}$$

and

$$|D^\alpha e_\mu(y_\mu)| \leq \frac{M}{r_\mu^{|\alpha|}} \qquad \text{for } |\alpha| \leq s_\mu. \tag{4.3}$$

For the unit vectors η_μ with $x_\mu + \eta_\mu \varrho_\mu \in \Omega_\mu$ for $0 < \varrho_\mu < r_\mu$ we have

$$\begin{aligned} u(x_1, x_2) &= -e_1(x_1 + \eta_1 \varrho_1) u(x_1 + \eta_1 \varrho_1, x_2)\Big|_{\varrho_1 = 0}^{\varrho_1 = r_1} \\ &= -\int_0^{r_1} \frac{\partial\big(e_1(x_1 + \eta_1 \varrho_1) u(x_1 + \eta_1 \varrho_1, x_2)\big)}{\partial \varrho_1}\, d\varrho_1, \end{aligned}$$

and by partial integration we get

$$u(x_1, x_2) = \frac{(-1)^{s_1}}{(s_1 - 1)!} \int_0^{r_1} \varrho_1^{s_1 - 1} \frac{\partial^{s_1}\big(e_1(x_1 + \eta_1 \varrho_1) u(x_1 + \eta_1 \varrho_1, x_2)\big)}{\partial \varrho_1^{s_1}}\, d\varrho_1.$$

By using e_2 and the Fubini theorem one analogously gets

$$\begin{aligned} u(x_1, x_2) = \frac{(-1)^{s_1 + s_2}}{(s_1 - 1)!(s_2 - 1)!} \int_0^{r_1} \int_0^{r_2} \varrho_1^{s_1 - 1} \varrho_2^{s_2 - 1} \frac{\partial^{s_1 + s_2}}{\partial \varrho_1^{s_1} \partial \varrho_2^{s_2}} \big[e_1(x_1 + \eta_1 \varrho_1) \\ \times e_2(x_2 + \eta_2 \varrho_2) u(x_1 + \eta_1 \varrho_1, x_2 + \eta_2 \varrho_2)\big]\, d\varrho_1 d\varrho_2. \end{aligned}$$

Integration over $\Sigma_1 \times \Sigma_2$, again the Fubini theorem and the Schwarz inequality yield with $C^{x_\mu} := x_\mu - C_{x_\mu}$

(4.4)

$$
\begin{aligned}
&|\Sigma_1 \times \Sigma_2|\,|u(x_1,x_2)| \\
&\quad = \frac{1}{(s_1-1)!(s_2-1)!}\left|\int_{C^{x_1}}\int_{C^{x_2}} \varrho_1^{s_1-n_1}\varrho_2^{s_2-n_2}\frac{\partial^{s_1+s_2}}{\partial\varrho_1^{s_1}\partial\varrho_2^{s_2}}[e_1e_2u]\,dy_1\,dy_2\right| \\
&\quad \le c\int_{C^{x_1}} \varrho_1^{s_1-n_1}\left|\int_{C^{x_2}} \varrho_2^{2(s_2-n_2)}\,dy_2\right|^{1/2}\left|\int_{C^{x_2}}\left|\frac{\partial^{s_1+s_2}}{\partial\varrho_1^{s_1}\partial\varrho_2^{s_2}}[e_1e_2u]\right|^2 dy_2\right|^{1/2} dy_1 \\
&\quad \le c\left|\int_{C^{x_1}} \varrho_1^{2(s_1-n_1)}\,dy_1\right|^{1/2}\left|\int_{C^{x_2}} \varrho_2^{2(s_2-n_2)}\,dy_2\right|^{1/2} \\
&\qquad \times \left|\int_{C^{x_1}}\int_{C^{x_2}}\left|\frac{\partial^{s_1+s_2}}{\partial\varrho_1^{s_1}\partial\varrho_2^{s_2}}[e_1e_2u]\right|^2 dy_1\,dy_2\right|^{1/2} \\
&\quad \le c\left|\int_{C^{x_1}} \varrho_1^{2(s_1-n_1)}\,dy_1\right|^{1/2}\left|\int_{C^{x_2}} \varrho_2^{2(s_2-n_2)}\,dy_2\right|^{1/2}\|e_1e_2u\|_{s_1,s_2;\Omega}.
\end{aligned}
$$

We have (cf. Wloka [4, Section 6])

$$
(4.5) \qquad \int_{C^{x_\mu}} \varrho_\mu^{2(s_\mu-n_\mu)}\,dy_\mu = \frac{2\pi^{n_\mu/2}}{\Gamma(n_\mu/2)(2s_\mu-n_\mu)}r_\mu^{2s_\mu-n_\mu}.
$$

If we apply the Leibniz product rule to e_1e_2u and refer back to (4.3) we obtain

$$
(4.6) \qquad \|e_1e_2u\|_{s_1,s_2;\Omega} \le c\cdot \max_{|\alpha_1|\le s_1} r_1^{-|\alpha_1|}\cdot \max_{|\alpha_2|\le s_2} r_2^{-|\alpha_2|}\cdot \|u\|_{s_1,s_2;\Omega}.
$$

Taking account of (4.5) and (4.6) in (4.4), we get

$$
|\Sigma_1\times\Sigma_2|\,|u(x_1,x_2)| \le c\cdot r_1^{s_1-n_1/2}\cdot \max_{|\alpha_1|\le s_1} r_1^{-|\alpha_1|}\cdot r_2^{s_2-n_2/2}\cdot \max_{|\alpha_2|\le s_2} r_2^{-|\alpha_2|}\cdot\|u\|_{s_1,s_2;\Omega},
$$

i.e. (4.2). □

Corollary 4.3. *For domains $\Omega_\mu \subset \mathbf{R}^{n_\mu}$ $(\mu = 1,2)$ satisfying the uniform cone condition and for $\Omega = \Omega_1\times\Omega_2$ we have the continuous embedding*

$$
H^{s_1,s_2}(\Omega) \hookrightarrow C^{t_1,t_2}(\overline{\Omega})
$$

for $s_\mu, t_\mu \in \mathbf{N}_0$ with $s_\mu - t_\mu > n_\mu/2$.

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References

[1] Doppel, K., and R. Hochmuth: On the regularity of solutions of a homogeneous Dirichlet problem for a non-hypoelliptic linear partial differential operator. - Ann. Acad. Sci. Fenn. Ser. A I Math. 16, 1991, 183–198.

[2] Grisvard, P.: Elliptic problems in nonsmooth domains. - Monographs and Studies in Mathematics 24, Pitman, Boston, 1985.

[3] Hochmuth, R.: Regularitätsresultate für ein Randwertproblem einer nicht-hypoelliptischen linearen partiellen Differentialgleichung. - Dissertation, Freie Universität Berlin, 1989.

[4] Wloka, J.: Partial differential equations. - Cambridge University Press, Cambridge, 1987.

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