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HARMONIC MORPHISMS BETWEEN SEMI-RIEMANNIAN MANIFOLDS

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Abstract. A smooth map $f\colon M \to N$ between semi-riemannian manifolds is called a harmonic morphism if f pulls back harmonic functions (i.e., local solutions of the Laplace–Beltrami equation) on N into harmonic functions on M. It is shown that a harmonic morphism is the same as a harmonic map which is moreover horizontally weakly conformal, these two notions being likewise carried over from the riemannian case. Additional characterizations of harmonic morphisms are given. The case where M and N have the same dimension n is studied in detail. When $n=2$ and the metrics on M and N are indefinite, the harmonic morphisms are characterized essentially by preserving characteristics.

Introduction

For maps between riemannian manifolds the notions of harmonic morphism and horizontally weakly conformal map were introduced and related to each other, via the well-known notion of harmonic map, in an earlier paper [1], and independently in Ishihara [10]. Recently the corresponding notions for *semi-riemannian manifolds* have been studied by Parmar [12]. We refer to O'Neill [11] concerning semi-riemannian manifolds (where the metric tensor may be indefinite, and hence the Laplace–Beltrami operator may not be elliptic).

The extension of the notion of *harmonic map* to the semi-riemannian case is straightforward (cf. the beginning of Section 3). In particular, we have the notion of a *harmonic function* on a semi-riemannian manifold, viz. a smooth function annihilated by the Laplace–Beltrami operator. A *harmonic morphism* between semi-riemannian manifolds M, N is defined as a smooth map $M \to N$ which pulls back local harmonic functions on N into local harmonic functions on M. We show that, like in the riemannian case [1], [10], a harmonic morphism is the same as a smooth map which is harmonic and *horizontally weakly conformal* (Theorem 3). This latter notion is essentially carried over from the riemannian case, except that the non-negative scalar λ in [10] (denoted by λ^2 in [1]) should now just be real-valued, cf. Section 2 below. Indeed, there exist (in the non-riemannian case) harmonic morphisms for which λ takes both positive and negative values (Examples 5.1 and 5.6).

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A major step towards the proof of the stated main result is the characterization of the notion of harmonic morphism by an apparently more restrictive pull-back property involving fully the Laplace–Beltrami operators Δ_M, Δ_N, rather than just the functions annihilated by them (Proposition 3.2). This characterization depends on the existence of solutions of the Laplace–Beltrami equation $\Delta_N v = 0$ near a given point of N where the partial derivatives of orders 1 and 2 of v in local coordinates are prescribed (Lemma 3.2). This local existence result was obtained by Ishihara [10] in the case of a riemannian manifold N; and it is easily established if N is either analytic or lorentzian. For an arbitrary semi-riemannian manifold N, Lemma 3.2 is contained in a local existence theorem due to Lars Hörmander (personal communication), and I am indebted to him for permission to include his proof of this theorem in the appendix.

Apart from the use of Lemma 3.2 our proof of the characterization of the notion of harmonic morphism in Theorem 3 is a combination of arguments given for the riemannian case in [1] and [2], drawing only on the most basic elements of semi-riemannian geometry. Alternatively, Ishihara's proof in [10] likewise seems to extend to the semi-riemannian case in view of Hörmander's result.

In Section 4 we characterize in detail, and more explicitly, the harmonic morphisms between semi-riemannian manifolds M, N of the same dimension. In particular, we extend to this setting most of the results obtained by Gehring and Haahti [4] for the case where $M = N = \mathbf{R}^n$ endowed with a constant semi-riemannian metric and where the maps in question are homeomorphisms.—For general semi-riemannian manifolds M, N with $\dim M > \dim N$ further results and examples are found in [12].

An interesting application of the notion of harmonic morphism in the semi-riemannian case to the riemannian case has been given quite recently by Gudmundsson [5], using the classical representation of hyperbolic m-space as the upper sheet of a hyperboloid in $(m+1)$-dimensional Minkowski space.

A number of references to the literature concerning harmonic morphisms in the riemannian case are found in [5].

1. Semi-riemannian manifolds

A semi-riemannian manifold M is a C^∞-manifold endowed with a metric tensor g_M, that is, a symmetric non-degenerate $(0,2)$ tensor field on M with constant indices of positivity and negativity $\mathrm{ind}_+ M$ and $\mathrm{ind}_- M$, respectively. The non-degeneracy means that $\mathrm{ind}_+ M + \mathrm{ind}_- M = \dim M$ (the dimension of M).

A subspace U of the tangent space $T_x(M)$, $x \in M$, is called *non-degenerate* if the restriction of g_M^x to $U \times U$ is non-degenerate, that is, if 0 is the only vector $X \in U$ such that $g_M^x(X,Y) = 0$ for every $Y \in U$; otherwise U is called degenerate.

Let $f\colon M \to N$ be a C^1-map between semi-riemannian manifolds M and N of dimensions m and n, respectively. For each $x \in M$ we consider the following two subspaces $K_x(f) = K_x$ and $K_x^\perp(f) = K_x^\perp$ of $T_x(M)$:

$$K_x = \ker df(x) = \{X \in T_x(M) \mid df(x)(X) = 0\},$$
$$K_x^\perp = \{X \in T_x(M) \mid g_M^x(X,Y) = 0 \text{ for every } Y \in K_x\}.$$

In terms of local coordinates $(y^1, \ldots, y^n)$ in N near $f(x)$ we have

$$K_x^\perp = \operatorname{span}\{\nabla_M f^1(x), \ldots, \nabla_M f^n(x)\}, \tag{1}$$

where $f^k = y^k \circ f$ denotes the kth component of f, and ∇_M is the gradient operator acting on scalar fields on M.

In the riemannian case we have $K_x \oplus K_x^\perp = T_x(M)$, and it is customary then to call K_x the *vertical* space and $K_x^\perp$ the *horizontal* space at x. In the semi-riemannian case, although of course $\dim K_x + \dim K_x^\perp = m$, it may occur that $K_x + K_x^\perp \neq T_x(M)$, or equivalently: $K_x \cap K_x^\perp \neq \{0\}$; this is further equivalent to K_x (or just as well $K_x^\perp$) being degenerate.—The following definition is adapted from [12]:

Definition 1. A C^1-map $f\colon M \to N$ between semi-riemannian manifolds M, N is called *non-degenerate* if $K_x(f)$ (or equivalently $K_x^\perp(f)$) is non-degenerate for every $x \in M$; otherwise f is called degenerate.

Clearly, if M is riemannian, every C^1-map $M \to N$ is non-degenerate. It is mainly in Section 4, where we treat rather completely the case $\dim M = \dim N$, that the notion of non-degenerate map comes into play.

2. Horizontally weakly conformal maps

When defining this notion in the semi-riemannian case we do not want to exclude maps which are degenerate (in the sense of Definition 1 above). We are led therefore to the following definition, justified by Lemma 2 and Theorem 3 below.

Definition 2. A C^1-map $f\colon M \to N$ between semi-riemannian manifolds M, N is called *horizontally weakly conformal* if

1° For any $x \in M$ at which K_x (or equivalently $K_x^\perp$) is non-degenerate and $df(x) \neq 0$, the restriction of $df(x)$ to $K_x^\perp$ is surjective, and conformal in the sense that there is a (necessarily unique) real number $\lambda(x) \neq 0$ such that

$$g_N^{f(x)}\big(df(X), df(Y)\big) = \lambda(x) g_M^x(X,Y) \text{ for every } X, Y \in K_x^\perp.$$

2° For any $x \in M$ at which K_x is degenerate we have $K_x^\perp \subset K_x$, that is,

$$g_M^x(X,Y) = 0 \text{ for every } X, Y \in K_x^\perp.$$

The *dilatation* λ of a horizontally weakly conformal map $f\colon M \to N$ is defined as a scalar on M in accordance with 1°, extended to all of M by $\lambda(x) = 0$ if $df(x) = 0$ or if K_x is degenerate. The term "weakly" refers to the possible occurrence of points $x \in M$ at which $\lambda(x) = 0$ according to this extension. (In [1], [2] a horizontally weakly conformal map between riemannian manifolds was termed a *semiconformal* map.)

Clearly $\dim M \geqslant \dim N$ if there exists a horizontally weakly conformal map $f\colon M \to N$ and a point $x \in M$ such that $\lambda(x) \neq 0$, that is: $df(x) \neq 0$ and K_x is non-degenerate. In fact, at such a point x the rank $r(x)$ of $df(x)$ equals $\dim N = n$, $df(x)$ being surjective. However, even for a *harmonic morphism* (cf. Theorem 3) there may be points $x \in M$ at which K_x and $K_x^\perp$ are degenerate, and hence $0 < r(x) < m$, noting that $r(x) = \dim K_x^\perp$. For arbitrary $m \geqslant 2$ and $n \geqslant 1$ there even exist semi-riemannian manifolds M, N with $\dim M = m$, $\dim N = n$ and harmonic morphisms $f\colon M \to N$ such that K_x is degenerate for *every* point $x \in M$ (see Examples 5.2 and 5.5).

Also note that, if there exists $x \in M$ such that $\lambda(x) > 0$, say, then we have $\operatorname{ind}_+ M \geqslant \operatorname{ind}_+ N$, $\operatorname{ind}_- M \geqslant \operatorname{ind}_- N$, with equality when $\dim M = \dim N$. In fact, any subspace of positivity [negativity] of $T_{f(x)}(N)$ is the bijective image under $df(x)$ of a subspace of positivity [negativity] of $K_x^\perp \subset T_x(M)$.

If f is non-degenerate (Definition 1) the possibility 2° in Definition 2 does not occur; this simplification therefore always applies when M is riemannian (in particular when $m = 1$). For a horizontally weakly conformal map $f\colon M \to N$ between riemannian manifolds we clearly have $\lambda(x) \geqslant 0$ for every $x \in M$ (and here we have in [1] defined the dilatation as the square root of the above λ, namely as the coefficient of conformality of the restriction of df to $K^\perp$). (The need for allowing also negative values of λ in the semi-riemannian case becomes apparent if one simply multiplies the metric on one of the manifolds by -1, a circumstance which seems to have been overlooked in [12].) See Remark 3.2 as to the possibility of a dilatation λ of variable sign in the non-riemannian case.

Lemma 2. *A C^1-map $f\colon M \to N$ is horizontally weakly conformal and has the dilatation λ if and only if*

$$g_M\big(\nabla_M(v \circ f), \nabla_M(w \circ f)\big) = \lambda\big[g_N(\nabla_N v, \nabla_N w) \circ f\big] \tag{2}$$

for every pair of C^1-functions v, w on N.

Remark 2. By polarization it suffices to consider pairs (v, w) with $v = w$. Clearly (2) extends to a local version in which v and w may be defined just in an open subset of N (with non-empty preimage). Next, it is enough to verify (2) for $v = y^k$, $w = y^l$, whereby $(y^1, \dots, y^n)$ are local coordinates in N; then (2) takes the form

$$g_M(\nabla_M f^k, \nabla_M f^l) = \lambda[g_N^{kl} \circ f] \tag{3}$$

for $k, l = 1, \dots, n$. It follows that the dilatation λ is continuous on M.

Proof of Lemma 2. 1° For any $x \in M$ at which K_x is non-degenerate the proof of [1, Lemma, p. 119] carries over mutatis mutandis.

2° For any $x \in M$ at which K_x is degenerate it appears from (1) that (3) holds at x with $\lambda(x) = 0$ if and only if $g_M^x(X,Y) = 0$ for every $X, Y \in K_x^{\perp}$. □

Clearly, when $n = 1$, *every* C^1-map $f\colon M \to N$ is horizontally weakly conformal. If $N = \mathbf{R}$ (with the standard metric) the dilatation of f is $\lambda = g_M(\nabla f, \nabla f)$.

3. Harmonic maps and harmonic morphisms

The Laplace–Beltrami operator Δ_M on a semi-riemannian manifold M is given in local coordinates x^i by

$$\Delta_M = \frac{1}{\sqrt{|g_M|}} \sum_{i=1}^{m} D_i \Big(\sqrt{|g_M|} \sum_{j=1}^{m} g_M^{ij} D_j \Big), \tag{4}$$

where $D_i = \partial/\partial x^i$ and $g_M = |\det(g_{ij}^M)|$, g_M^{ij} and g_{ij}^M being the contravariant and the covariant components of the metric tensor g_M. The operator Δ_M is not elliptic when $\operatorname{ind}_+ M > 0$ and $\operatorname{ind}_- M > 0$. Nevertheless we shall keep the term *harmonic functions* for C^2-smooth local solutions to the Laplace–Beltrami equation $\Delta_M h = 0$.

Like in the riemannian case, the *tension field* $\tau(f)$ of a C^2-map $f\colon M \to N$ between semi-riemannian manifolds is defined as the vector field along f which to each point $x \in M$ assigns the tangent vector, denoted $\tau(f)(x)$ ($\in T_{f(x)}(N)$), whose contravariant components $\tau^k(f)(x)$ in terms of local coordinates $(y^1, \dots, y^n)$ in N are defined by

$$\tau^k(f) = \Delta_M f^k + \sum_{\alpha,\beta=1}^{n} g_M(\nabla f^\alpha, \nabla f^\beta)(\Gamma_{\alpha\beta}^k \circ f).$$

Here the $\Gamma_{\alpha\beta}^k$ denote the Christoffel symbols for the target manifold N. If $\tau(f) = 0$, f is called a *harmonic map*.

For any C^2-function $f\colon M \to \mathbf{R}$ we have $\tau(f) = \Delta_M f$; and f is therefore a harmonic map if and only if f is a harmonic function, or equivalently: a harmonic morphism (see Definition 3 below). This extends to general target manifolds N as follows.

Lemma 3.1. *Let $f\colon M \to N$ be a horizontally weakly conformal C^2-map with dilatation λ. The tension field $\tau(f)$ is then given in terms of local coordinates (y^k) in N by*

$$\tau^k(f) = \Delta_M f^k - \lambda\big[(\Delta_N y^k) \circ f\big].$$

In particular, $\tau^k(f) = \Delta_M f^k$ if the local coordinates y^k in N are chosen as harmonic functions.

Proof. As in the riemannian case [1, p. 123] the identity map id: $N \to N$ is harmonic:

$$\tau^k(\mathrm{id}) = \Delta_N y^k + \sum_{\alpha,\beta=1}^{n} g_N^{\alpha\beta}\Gamma_{\alpha\beta}^k = 0$$

(the former equation being a special case of the definition of $\tau^k(f)$ above). It remains to compose with f, multiply by λ, and apply (3) in Remark 2. ▫

In connection with the second assertion of Lemma 3.1, note that *every semi-riemannian manifold admits local coordinates which are harmonic functions.* This is an easy consequence of the following lemma.

Lemma 3.2. *For any point $q \in N$ and any C^2-function v on N satisfying $\Delta_N v(q) = 0$ there exists, for given $s \geqslant 2$, a harmonic C^s-function h in some open neighbourhood V of q such that h and v have the same partial derivatives at q of orders 1 and 2 with respect to local coordinates in N near q.*

Equivalently, for any $q \in N$ there exists in an open neighbourhood of q a harmonic C^s-function h which, in terms of local coordinates $(y^1, \dots, y^n)$ centred at q, has arbitrarily prescribed values of $D_k h(0)$ and $D_k D_l h(0)$ $(k, l = 1, \dots, n)$ compatible with $\Delta_N h(q) = 0$.

For a general semi-riemannian C^∞-manifold Lemma 3.2 is due to Hörmander (personal communication), and his proof of a more general result is given in the appendix. According to the remark there one may even take $s = \infty$ in Lemma 3.2.

For a *riemannian* manifold, Lemma 3.2 is due to Ishihara [10], who applied it to prove Theorem 3 below, independently and in a different way from [1] (where instead a general result from potential theory was used to show that every harmonic morphism has the property c_+ in Proposition 3.3 below).

For an *analytic* semi-riemannian manifold the lemma is easily obtained by use of the Cauchy–Kovalevsky theorem (see e.g. [8, p. 119]), applied after choosing local coordinates $(y^1, \dots, y^n)$ centred at q so that $g^{nn} \neq 0$ at $q = 0$. Replacing v as a function of $(y^1, \dots, y^n)$ by its Taylor polynomial of degree $\leqslant 2$ expanded from 0 we see that v can be taken to be analytic from the outset. There exists then in a neighbourhood of $q = 0$ an analytic solution h of $\Delta_N h = 0$ such that

$$h = v, \qquad D_n h = D_n v \qquad \text{when } y^n = 0.$$

Applying D_k and $D_k D_l$ with $k, l < n$ we obtain for $y^n = 0$, and in particular at $q = 0$,

$$D_k h = D_k v, \qquad D_k D_n h = D_k D_n v, \qquad D_k D_l h = D_k D_l v, \qquad (k, l < n).$$

The remaining equality $D_n^2 h(0) = D_n^2 v(0)$ follows now from $\Delta_N h(0) = 0 = \Delta_N v(0)$.

There is a similar short proof in the *lorentzian* case, defined by $\operatorname{ind}_+ N = 1$ or $\operatorname{ind}_- N = 1$, using now the local solvability of the Cauchy problem for the hyperbolic equation $\Delta_N h = 0$ with the above initial data, $\{y \mid y^n = 0\}$ being chosen spacelike (see Hadamard [6]).

Definition 3. A C^2-map $f\colon M \to N$ between semi-riemannian manifolds is called a *harmonic morphism* if, for any harmonic function v in an open set $V \subset N$ such that $f^{-1}(V) \neq \emptyset$, the pull-back $v \circ f$ is harmonic in $f^{-1}(V)$.

The notion of harmonic morphism remains the same if solely C^∞-solutions v of $\Delta_N v = 0$ (in V) are considered in the above definition. This follows from the proof of (c) $\Rightarrow$ (b) in Proposition 3.2 in view of Lemma 3.2 above and a comment shortly thereafter.

Remark 3.1. Clearly, if $f\colon M \to N$ and $g\colon N \to P$ are harmonic morphisms, then so is $g \circ f\colon M \to P$. As to the inverse of an injective harmonic morphism, see Theorem 4.3 below.

Theorem 3. *A harmonic morphism is the same as a harmonic map which is horizontally weakly conformal.*

This main result is contained in the combined Propositions 3.1 and 3.2 below. In view of Lemma 3.1 the theorem can also be formulated as follows: a harmonic morphism is the same as a horizontally weakly conformal map $f\colon M \to N$ whose components $f^k = y^k \circ f$ in terms of harmonic local coordinates y^k in N are harmonic in M (cf. the paragraph just before Lemma 3.2).

Proposition 3.1. *A C^2-map $f\colon M \to N$ is harmonic and horizontally weakly conformal if and only if there exists a scalar λ on M such that*

$$\Delta_M(v \circ f) = \lambda\big[(\Delta_N v) \circ f\big] \tag{5}$$

for every C^2-function v on N. For such a map f the scalar λ equals the dilatation of f, cf. Definition 2.

Proof. The proof for the riemannian case given in [1, p. 124] carries over in view of the first part of Lemma 3.1 together with (3) in Remark 2. □

We proceed to characterize harmonic morphisms by two apparently stronger pull-back properties (the former being (5) in Proposition 3.1 above).

Proposition 3.2. *For a C^2-map $f\colon M \to N$ the following are equivalent:*

(a) *There exists a scalar λ on M such that, for any C^2-function v on N,*
$$\Delta_M(v \circ f) = \lambda\big(\Delta_N v) \circ f\big].$$

(b) *For any point $p \in M$ and any C^2-function v on N we have, writing $q = f(p)$, the implication*
$$\Delta_N v(q) = 0 \qquad \Rightarrow \qquad \Delta_M(v \circ f)(p) = 0.$$

(c) *f is a harmonic morphism. Explicitly: for any harmonic function v in an open set $V \subset N$, $v \circ f$ is harmonic in $f^{-1}(V)$ (assumed non-empty).*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) is obvious because (c) reduces to the case where v extends to a C^2-function on all of N.

(b) $\Rightarrow$ (a). Adapting the simple argument in [2, p. 129] we consider a C^2-function v on N, and write $q = f(p)$ for given $p \in M$. Choose a C^2-function w on N so that $\Delta_N w(q) \neq 0$. (In terms of local coordinates $(y^1, \dots, y^n)$ in N centred at q we have for example $\Delta_N(y^k y^l)(0) = 2g_N^{kl}(0) \neq 0$ for some k, l.) Write

$$t = \Delta_N v(q)/\Delta_N w(q),$$

and apply (b) to $v - tw$ in place of v, noting that $\Delta_N(v - tw)(q) = 0$. This gives

$$\Delta_M(v \circ f)(p) = t\Delta_M(w \circ f)(p) = \lambda(p)\Delta_N v(q)$$

with $\lambda(p) = \Delta_M(w \circ f)(p)/\Delta_N w(q)$.

(c) $\Rightarrow$ (b) follows by application of Lemma 3.2 in view of the following identity (in which $D_k = \partial/\partial y_k$, etc.):

$$\Delta_M(v \circ f) = g_M(\nabla f^k, \nabla f^l)\big[(D_k D_l v) \circ f\big] + (\Delta_M f^k)\big[(D_k v) \circ f\big],$$

cf. [1, (17), p. 124], which shows that $\Delta_M(v \circ f)(p) = \Delta_M(h \circ f)(p) = 0$. □

Having thus completed the proof of Theorem 3 we now bring a result similar to Proposition 3.2 for the case where $\lambda \geqslant 0$ in (a), thereby characterizing the harmonic morphisms with non-negative dilatation.

Proposition 3.3. *For a C^2-map $f\colon M \to N$ the following are equivalent:*

a_+ *There exists a scalar $\lambda \geqslant 0$ on M such that, for any C^2-function v on N,*

$$\Delta_M(v \circ f) = \lambda\big[(\Delta_N v) \circ f\big].$$

b_+ *For any point $p \in M$ and any C^2-function v on N we have, writing $q = f(p)$, the implication*

$$\Delta_N v(q) \geqslant 0 \qquad \Rightarrow \qquad \Delta_M(v \circ f)(p) \geqslant 0.$$

c_+ *For any C^2-function v in an open set $V \subset N$ such that $f^{-1}(V) \neq \emptyset$ we have the implication*

$$\Delta_N v \geqslant 0 \text{ in } V \qquad \Rightarrow \qquad \Delta_M(v \circ f) \geqslant 0 \text{ in } f^{-1}(V).$$

Proof. (Note that Lemma 3.2 is not used here.) $a_+ \Rightarrow b_+ \Rightarrow c_+$ is obvious.

$b_+ \Rightarrow a_+$ is established just like (b) $\Rightarrow$ (a) in the proof of Proposition 3.2.

$c_+ \Rightarrow b_+$: Choose a C^2-function w on N so that $\Delta_N w(q) > 0$. For any $\varepsilon > 0$ we then have $\Delta_N(v + \varepsilon w) > 0$ in some open neighbourhood V_ε of q, and hence, by c_+, $\Delta_M\big[(v + \varepsilon w) \circ f\big] \geqslant 0$ in $f^{-1}(V_\varepsilon)$, in particular at p. For $\varepsilon \to 0$ this leads to $\Delta_M(v \circ f)(p) \geqslant 0$. □

Remark 3.2. A similar characterization of harmonic morphisms with dilatation $\lambda \leqslant 0$ is of course obtained by reversing one of the inequality signs in b_+ and in c_+. For non-riemannian (connected) M, N, however, it can occur that a harmonic morphism (even a non-degenerate one) has dilatation λ taking both signs, cf. Examples 5.1 and 5.6. This phenomenon does not occur when $\dim M = \dim N > 2$ (see (b) in Theorem 4.1 below), or when $\dim M = \dim N = 2$ and f is non-degenerate (see the final assertion of Theorem 4.2, noting that, in view of the third paragraph after Definition 2, M and N must either be both lorentzian or both riemannian, possibly after multiplying both metrics by -1).

4. The case $\dim M = \dim N$

In this case we shall use the term *weakly conformal* as synonymous with horizontally weakly conformal in the sense of Definition 2, noting that $df(x)\colon T_x(M) \to T_{f(x)}(N)$ is now conformal if surjective (hence bijective); i.e. in the situation 1° in Definition 2.

As a preparation to the 2-dimensional case consider a 2-dimensional semi-riemannian manifold M which is *lorentzian*: $\operatorname{ind}_- M = \operatorname{ind}_+ M = 1$. A *characteristic*, or zero-line, of M is a connected 1-dimensional immersed submanifold L of M such that the tangent to L at any point $x \in L$ is one of the characteristic subspaces of $T_x(M)$, i.e. those two 1-dimensional subspaces on which the restriction of g_M^x is 0. A *characteristic chart*, or characteristic coordinate system, in M is a C^∞-diffeomorphism π of an open subset U of M onto the product of two non-empty intervals $I_1, I_2 \subset \mathbf{R}$ such that $\pi^{-1}(\{x^1\} \times I_2)$ and $\pi^{-1}(I_1 \times \{x^2\})$ are characteristics in U for any $x^1 \in I_1$, respectively $x^2 \in I_2$. (They are then the only characteristics in U which are maximal w.r.t. inclusion.) Equivalently, there should be a C^∞-function a on $I_1 \times I_2$ with values $a(x^1, x^2) \neq 0$ such that π becomes an isometry of U (with the metric inherited from g_M) onto $I_1 \times I_2$ endowed with the Lorentz metric

$$h_x(X, X) = 2a(x)^{-1} X^1 X^2. \tag{6}$$

The Laplace–Beltrami operator (4) for $I_1 \times I_2$ with the metric (6) reduces to

$$\Delta = 2aD_1D_2 \tag{7}$$

with $D_i = \partial/\partial x^i$, cf. the similar case $m = 2$ in Example 5.5. The harmonic functions are therefore the C^2-functions of the form

$$u(x) = \varphi(x^1) + \psi(x^2), \qquad (x^1, x^2) \in I_1 \times I_2. \tag{8}$$

The domain U of a characterisctic chart $\pi\colon U \to I_1 \times I_2$ is called a *characteristic patch.* Any 2-dimensional Lorentz manifold can be covered by characteristic patches, cf. e.g. [3, p. 62].

Theorem 4.1. *Let $f\colon M \to N$ be a C^2-map between connected semi-riemannian manifolds of equal dimension n.*

(a) *In the case $n = 2$: f is a harmonic morphism if and only if f is weakly conformal.*

(b) *In the case $n > 2$: if f is a harmonic morphism then f is weakly conformal with constant dilatation; and as a partial converse: if f is non-degenerate and weakly conformal with constant dilatation then f is a harmonic morphism.*

Proof. Consider first the case where $df(x)\colon T_x(M) \to T_{f(x)}(N)$ is bijective for every $x \in M$. In view of Proposition 3.2 and (3) in Remark 2, the proof of (a) and (b) in this situation is the same as in the riemannian case, see [1, p. 125 f].

Ad (a) in the general case. If f is a harmonic morphism then f is weakly conformal by Theorem 3.1. Conversely suppose that f is weakly conformal, with dilatation λ. We propose to verify (5) in Proposition 3.1. By continuity it is enough to show that (5) holds in a dense subset of M. Write $r(x) = \operatorname{rk} df(x)$, the rank of df at x, and

$$M_r = \{x \in M \mid r(x) = r\}, \qquad r = 0, 1, 2.$$

Then $M = M_0 \cup M_1 \cup M_2$, M_0 is closed, M_2 is open, and M_1 is open relatively to $M \setminus M_2$. It follows that the union of M_2 and the interiors of M_0 and M_1 is dense in M. It suffices therefore to verify (5) in each of these 3 open subsets of M. In M_2 this is just the particular case of (a) considered in the beginning of the proof. In $\operatorname{int} M_0$, the interior of M_0, f is locally constant, and so (5) holds there with $\lambda = 0$. It remains to consider f in any component of $\operatorname{int} M_1$, and we may therefore assume that

$$r(x) = 1 \text{ for every } x \in M. \tag{9}$$

(This situation can actually occur, cf. Example 5.2.) It follows in view of Definition 2 that K_x and $K_x^\perp$ are degenerate and therefore equal:

$$K_x = K_x^\perp \tag{10}$$

because $\dim M = 2$. Hence g_M^x must be indefinite: $\operatorname{ind}_+ M = \operatorname{ind}_- M = 1$. Given $p \in M$ choose local coordinates (y^1, y^2) in an open neighbourhood V of $f(p)$ in N and characteristic local coordinates (x^1, x^2) in some characteristic patch $U = \pi^{-1}(I_1 \times I_2) \subset M$ such that $p \in U \subset f^{-1}(V)$. In terms of these local coordinates the jacobian Df is of rank

$$\operatorname{rk}\begin{pmatrix} D_1 f^1, & D_2 f^1 \\ D_1 f^2, & D_2 f^2 \end{pmatrix} = 1 \tag{11}$$

according to (9). We may also assume that e.g. $D_1 f^1(p) \neq 0$, and further that

$$D_1 f^1(x) \neq 0 \tag{12}$$

for all $x \in U$. It follows that

$$K_x = \{(X^1, X^2) \mid D_1 f^1(x) X^1 + D_2 f^1(x) X^2 = 0\}$$

because this relation between X^1 and X^2 implies $D_1 f^2(x) X^1 + D_2 f^2(x) X^2 = 0$ in view of (11), (12). Hence the contravariant vector $X := (-D_2 f^1(x), D_1 f^1(x))$ spans $K_x = K_x^\perp$, cf. (10), and so $-2 D_2 f^1(x) D_1 f^1(x) = a(x) h_x(X, X) = 0$ in view of (6). Using (12), (11), we infer that

$$D_2 f^1 = D_2 f^2 = 0$$

in U, that is, $f^1(x)$ and $f^2(x)$ depend only on x^1. So does therefore $v \circ f$ for any C^2-function v on N, and we conclude that indeed $\Delta_M(v \circ f) = 0$ on account of (7).

Ad (b) in the general case. If f is a harmonic morphism then f is weakly conformal by Theorem 3. In proving that the dilatation λ of f is locally constant and hence constant, we note that M is second countable. According to the beginning of the present proof, λ is constant and $\neq 0$ in every component of the open set M' of all points of M at which $df(x)$ is bijective. Moreover, $\lambda = 0$ in $M \setminus M'$. The range of the continuous function λ is therefore a non-empty countable connected subset of $\mathbf{R}$, i.e. a singleton, and so λ is constant.—As to the converse statement in the remaining case where $\lambda = 0$ and f is non-degenerate (Definition 1), we infer from 1° in Definition 2 that $df(x) = 0$ for every $x \in M$, and so f is *constant*, in particular a harmonic morphism. □

Remark 4.1. The requirement in the second assertion in (b) of Theorem 4.1 that f be non-degenerate is automatically fulfilled if $\lambda \neq 0$, but cannot be dropped in the case $\lambda = 0$, cf. Example 5.5 with $m = n \geqslant 3$.—Gehring and Haahti [4] obtained Theorem 4.1 for non-degenerate homeomorphisms of $\mathbf{R}^n$ endowed with a constant semi-riemannian metric, cf. Example 5.3 below.

It is well known that a harmonic morphism between Riemann surfaces is the same as a holomorphic or antiholomorphic map, cf. e.g. [1, p. 127]. We shall now give an analogous explicit characterization of the harmonic morphisms, or weakly conformal C^2-maps, between 2-dimensional Lorentz manifolds M, N. In place of $\pm$holomorphy enters primarily a property of carrying characteristics into characteristics (see the second paragraph of the present section).

Theorem 4.2. *Let $f \colon M \to N$ be a C^2-map between connected 2-dimensional Lorentz manifolds M, N. Then f is a harmonic morphism if and only if M*

can be covered by characteristic patches U, each having one of the following three properties labelled according to the value of $r(U) := \max\{\mathrm{rk}(df(x)) \mid x \in U\}$:

$0°$ $r(U) = 0$, *that is, f is constant in U.*

$1°$ $r(U) = 1$ *and f is constant along every characteristic of U from one of the two families.*

$2°$ $r(U) = 2$ *and f maps U into some characteristic patch V of N in such a way that every characteristic of U is mapped into some characteristic of V.*

If f is a non-degenerate, non-constant harmonic morphism then only Property $2°$ occurs, and f is conformal because $\mathrm{rk}(df(x)) = 2$ for every $x \in M$, that is, the dilatation λ of f has constant sign: $\lambda \lessgtr 0$.

It is understood in this theorem that different patches U from the stated covering of M may not have the same property $0°$, $1°$, or $2°$. In $2°$ it is understood that two characteristics of U from the same family, respectively from different families, are mapped onto subsets of two characteristics of V, likewise from the same family, respectively from different families.

Proof. Let $\mathscr{U}$ denote any covering of M by characteristic patches U such that $f(U)$ is contained in some characteristic patch V of N. Using characteristic coordinates (x^1, x^2) in U and (y^1, y^2) in V we may assume that

$$U = I_1 \times I_2, \qquad V = J_1 \times J_2,$$

where I_1, I_2, J_1, J_2 are open intervals of $\mathbf{R}$, and that the metric tensors on U and V have the following matrices (contravariant components)

$$(g_U^{ij}) = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, \qquad (g_V^{kl}) = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} \tag{13}$$

for certain C^∞-functions a on U and b on V (not taking the value 0), cf. (6). According to (7),

$$\Delta_U = 2a\frac{\partial^2}{\partial x^1 \partial x^2}, \qquad \Delta_V = 2b\frac{\partial^2}{\partial y^1 \partial y^2}. \tag{14}$$

Suppose first that $f\colon M \to N$ is a harmonic morphism. Since $\Delta_V y^k = 0$, $k = 1, 2$, we have from Definition 3: $\Delta_U f^k = 0$, and so, as in (8),

$$f^k(x^1, x^2) = \varphi_k(x^1) + \psi_k(x^2), \qquad k = 1, 2, \tag{15}$$

$$\det(Df) = \varphi_1'(x^1)\psi_2'(x^2) - \varphi_2'(x^1)\psi_1'(x^2), \tag{16}$$

where $\varphi_k \in C^2(I_1)$, $\psi_k \in C^2(I_2)$. From $\Delta_V[(y^k)^2] = 0$ we obtain $\Delta_U[(f^k)^2] = 0$, which easily leads to $\Delta_U[\varphi_k(x^1)\psi_k(x^2)] = 0$, that is, by (14),

$$\varphi_k'(x^1)\psi_k'(x^2) = 0, \qquad k = 1, 2. \tag{17}$$

If $r(U) \geqslant 1$ we may assume for example that $\varphi_1' \not\equiv 0$ and so, by (17), ψ_1 is constant.

In case $r(U) = 1$, we have $\det(Df) \equiv 0$ and hence ψ_2 is constant, by (16). In view of (15), f therefore has the form

$$f(x^1, x^2) = \bigl(\varphi(x^1), \psi(x^1)\bigr) \tag{18}$$

with φ or ψ (or both) non-constant. This amounts to f being constant along every characteristic in U of the form $x^1 =$ constant, without f being constant in all of U. It follows that f is *degenerate* because df has rank 1 at points $x = (x^1, x^2)$ with $\varphi'(x^1) \neq 0$ or $\psi'(x^1) \neq 0$.

In the principal case $r(U) = 2$ we have $\det(Df) \not\equiv 0$, hence from (16) $\psi_2' \not\equiv 0$ (because $\psi_1' \equiv 0$). In view of (17) and (15) this shows that φ_2 is constant and that f takes the form

$$f(x^1, x^2) = \bigl(\varphi(x^1), \psi(x^2)\bigr) \tag{19}$$

with φ and ψ non-constant. This amounts to f carrying every characteristic of U of the form $x^j =$ constant onto an arc (not necessarily open) of some characteristic of V of the form $y^j =$ constant, $j = 1, 2$.

Conversely, it is immediately seen that every C^2-map $f\colon I_1 \times I_2 \to J_1 \times J_2$ of the form (18) or (19) is a non-constant harmonic morphism with $r(U) = 1$ or 2, respectively, in view of (8).

Suppose finally that $f\colon M \to N$ is a *non-degenerate* (Definition 1) and non-constant harmonic morphism, and consider any patch $U \in \mathscr{U}$ (see above) such that $r(U) = 2$. Then

$$\operatorname{rk}\bigl(df(x)\bigr) = 2 \qquad \text{for every } x = (x^1, x^2) \in U. \tag{20}$$

For suppose e.g. that, in (19), $\varphi'(x^1) = 0$ for some $x^1 \in I_1$. Choosing $x^2 \in I_2$ so that $\psi'(x^2) \neq 0$ we find that $\operatorname{rk}(df) = 1$ at $x = (x^1, x^2)$, and this contradicts f being non-degenerate and weakly conformal, so that 1° in Definition 2 applies. In view of (20) a patch $U \in \mathscr{U}$ with $r(U) = 2$ cannot meet a patch $U_0 \in \mathscr{U}$ with $r(U_0) = 0$, that is, $\operatorname{rk}(df) \equiv 0$ in U_0. Because f is non-degenerate there are no patches $U_1 \in \mathscr{U}$ with $r(U_1) = 1$. Since M is connected we therefore conclude that all patches $U \in \mathscr{U}$ have $r(U) = 2$, the alternative being that f should be constant on every patch from $\mathscr{U}$ and hence in all of M, against hypothesis. □

The dilatation λ of the harmonic morphism $f\colon U \to V$ from (19) is easily found from (3) (with $(k, l) = (1, 2)$) or from (5) to be

$$\lambda(x) = \frac{a(x)}{b\bigl(f(x)\bigr)} \varphi'(x^1)\psi'(x^2), \tag{21}$$

while of course $\lambda \equiv 0$ in U in the cases $r(U) = 0$ or 1.

If a connected 2-dimensional Lorentz manifold M is *orientable* (and only in that case) it is possible to distinguish globally between the two characteristic 1-dimensional subspaces Z_x^1, Z_x^2 of the tangent space $T_x(M)$. Indeed, after fixing an orientation of M and hence of each $T_x(M)$, we may require that Z_x^1, when turned around 0 in $T_x(M)$ in the positive sense, becomes a subspace of *positivity* for the quadratic form g_M^x before it reaches Z_x^2. Each characteristic on M extends uniquely to a characteristic which is maximal w.r.t. inclusion. We have then two families of maximal characteristics, and either family covers M disjointly. From Theorems 4.1 and 4.2 we therefore obtain the following corollary, in which (c) is understood in accordance with the explanation concerning 2° in Theorem 4.2.

Corollary. *The following are equivalent for a non-degenerate, non-constant C^2-map $f\colon M \to N$ between connected orientable 2-dimensional Lorentz manifolds M, N:*

(a) *f is a harmonic morphism.*
(b) *f is weakly conformal (necessarily conformal, i.e. with dilatation $\lambda \lessgtr 0$).*
(c) *f maps every characteristic of M into some characteristic of N (necessarily onto an open arc of this characteristic of N).*

Remark 4.2. The assumption of non-degeneracy in the last assertion of Theorem 4.2 and in the above corollary is essential in order to exclude for example a harmonic morphism like (18) above, where $\lambda \equiv 0$ and f maps every characteristic $x^1 =$ constant onto a single point, cf. also Examples 5.2 and 5.4.

As a further application of the preceding results we finally consider *injective* harmonic morphisms, allowing a priori that $\dim M \neq \dim N$.

Theorem 4.3. *Let $f\colon M \to N$ be an injective harmonic morphism between semi-riemannian manifolds. Then $\dim M = \dim N$ and f is an open map. Moreover, $f^{-1}\colon f(M) \to M$ is a harmonic morphism if and only if f is non-degenerate. If $\dim M > 2$, f is always non-degenerate.*

Proof. We may assume that M and N are connected. As usual write $\dim M = m$, $\dim N = n$. With $r := \max\{\operatorname{rk}(df(x)) \mid x \in M\}$, the non-empty set $M_r = \{x \in M \mid \operatorname{rk}(df(x)) = r\}$ is open, and every point x in M_r has an open neighbourhood $U \subset M_r$ such that the restriction of f to U is a submersion of U onto an r-dimensional submanifold of N. When f is injective this requires $r = m$ (> 0), and consequently $r = n$ by Definition 2, noting that $K_x = \{0\}$ is non-degenerate for every $x \in M_r$. Thus $m = n = r$.

If $n = 2$ we may assume that M and N are either both riemannian or both lorentzian, cf. Remark 3.2. In the riemannian case the assertions (for $n = 2$) are known from [1, Corollary, p. 127], according to which a harmonic morphism is the same as a $\pm$holomorphic function. In the lorentzian case it follows from the proof of Theorem 4.2 above, in the case of a characteristic patch $U \in \mathscr{U}$ with $r(U) = 2$, that $\varphi\colon I_1 \to J_1$ and $\psi\colon I_2 \to J_2$ in (19) are injective, hence strictly monotone, because $f\colon I_1 \times I_2 \to J_1 \times J_2$ is injective. This shows that f is open, and that

$$f^{-1}(y^1, y^2) = \left(\varphi^{-1}(y^1), \psi^{-1}(y^2)\right)$$

for $(y^1, y^2) \in f(I_1 \times I_2) = \varphi(I_1) \times \psi(I_2)$. If f is non-degenerate then φ' and ψ' do not take the value 0, hence $df(x)$ is bijective for every $x \in M$, and it follows then that f^{-1} is a C^2-map and a harmonic morphism. Conversely, if f^{-1} is a harmonic morphism, or just a C^1-map, then $df(x)$ is bijective for every $x \in M$, and hence f is non-degenerate.

If $n > 2$ we apply (b) of Theorem 4.1. The constant dilatation λ of f is $\neq 0$, being non-zero at the points of the non-empty set M_r $(= M_n)$. Hence f is conformal, and $df(x)$ is bijective for every $x \in M$. This implies that f is a non-degenerate open map. It follows that f^{-1} is a conformal C^2-map with constant dilatation $1/\lambda \neq 0$, hence f^{-1} is non-degenerate and a harmonic morphism. □

Remark 4.3. An injective harmonic morphism between 2-dimensional Lorentz manifolds may well be degenerate, see Example 5.3.—For the case where $M = N = \mathbf{R}^n$ with a constant semi-riemannian metric, part of Theorem 4.3 is stated at the end of [4].

Remark 4.4. Every non-constant harmonic morphism between connected riemannian manifolds is an open map, [1, Theorem, p. 136], [2]. This does not extend to the semi-riemannian situation, cf. Example 5.4 and the second case in Example 5.6.

5. Examples

We shall use subscripts rather than superscripts for coordinates of points or contravariant vectors. In Examples 5.1 through 5.4 below the manifold M will be the plane $\mathbf{R}^2$ endowed with the constant indefinite metric $g_M^x(X, X) = 4X_1X_2$, hence

$$g_M^x(X, Y) = 2(X_1Y_2 + X_2Y_1),$$

where $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ range over $\mathbf{R}^2 = T_x(M)$, $x \in M$. Given two C^2-functions $\varphi, \psi\colon \mathbf{R} \to \mathbf{R}$ we then consider the C^2-maps

$$f\colon M \to M \text{ given by } f(x_1, x_2) = (\varphi(x_1), \psi(x_2)), \tag{22}$$

$$h\colon M \to \mathbf{R} \text{ given by } h(x_1, x_2) = \varphi(x_1) + \psi(x_2), \tag{23}$$

where $\mathbf{R}$ is endowed with the standard metric, and so the harmonic functions in (intervals of) $\mathbf{R}$ are the affine-linear functions. Clearly f and h are *harmonic morphisms*, both with dilatation

$$\lambda(x) = \varphi'(x_1)\psi'(x_2), \tag{24}$$

cf. also (19), (21) as to f. In all the examples below except Example 5.6 the maps are *degenerate* (Definition 1).

Example 5.1. With $\varphi(x_1) = x_1^2$, $\psi(x_2) = x_2$, the dilatation of f and h in (22), (23) is, by (24), $\lambda(x) = 2x_1$ which has *variable sign.* Note that f and h are degenerate maps because $K_x(f)$ and $K_x(h)$ are degenerate subspaces of $T_x(M)$ for $x = (0, x_2)$. Cf. Remark 3.2.

Example 5.2. With $\varphi(x_1) = x_1$, $\psi(x_2) = 0$, $K_x(f)$ becomes degenerate at *every* point $x \in M$ because the rank of $df(x)$ equals 1. An extension of this example to arbitrary dimensions $m \geqslant 2$ and n is given in Example 5.5 below with $a = 1$.

Example 5.3. With $\varphi(x_1) = x_1^3$, $\psi(x_2) = x_2$, f becomes a *homeomorphism* of $\mathbf{R}^2$. The inverse map is not smooth and hence not a harmonic morphism. This is because f is degenerate, cf. Theorem 4.3. Indeed, at the points $x = (0, x_2)$ we have $\lambda(x) = 0$, by (24), but $df(x) \neq 0$. (The fact that this latter phenomenon can occur seems to have been overlooked in [4].)

Example 5.4. With $\varphi(x_1) = x_1^3$ for $x_1 \geqslant 0$, $\varphi(x_1) = 0$ for $x_1 \leqslant 0$, and $\psi(x_2) = 0$ for all $x_2 \in \mathbf{R}$, the harmonic morphisms f and h are non-constant, degenerate, and non-open maps, being constant in the open half-plane $x_1 < 0$. Cf. Remark 4.4.

Example 5.5. For $m \geqslant 2$ and a given C^∞-function $a\colon \mathbf{R}^m \to \mathbf{R}_+$ we take $M = \mathbf{R}^m$ endowed with the following Lorentz metric:

$$g_M^x(X,Y) = \frac{1}{a(x)}\Big(-X_1Y_1 + \sum_{j=2}^m X_jY_j\Big), \qquad X, Y \in \mathbf{R}^m,$$

and we take $N = \mathbf{R}^n$, $n \geqslant 1$, with any constant semi-riemannian metric. Then (4) leads to

$$(25) \qquad \Delta_M = a\Big(-D_1^2 + \sum_{j=2}^m D_j^2\Big) - \frac{m-2}{2}\Big(-(D_1a)D_1 + \sum_{j=2}^m (D_ja)D_j\Big).$$

Define $f\colon M \to N$ by

$$f(x) = (x_1 + x_2, 0, \dots, 0) \quad (\in \mathbf{R}^n) \qquad \text{for } x \in M.$$

The covariant components of $\nabla_M f_1$ are $(1, 1, 0, \dots, 0)$ ($\in \mathbf{R}^m$), and we therefore have $g_M^x(\nabla_M f_1, \nabla_M f_1) = 0$. In view of (1) in Section 1 this shows that $K_x^\perp = \operatorname{span}\{\nabla_M f_1(x)\}$ is degenerate for every $x \in M$, and $g_M^x(X,Y) = 0$ for every $X, Y \in K_x^\perp$. Consequently, f is degenerate and horizontally weakly conformal with the constant dilatation $\lambda = 0$ (Case 2° in Definition 2 for every $x \in M$).

If $m = 2$ or if a is constant then f is a harmonic morphism since $\Delta_M(v \circ f) = 0$ for any C^2-function v on N, $v \circ f$ depending only on $x_1 + x_2$. A similar example is given in [12].

If $m \geqslant 3$ and e.g. $a(x) = \exp x_1$ then f is *not* a harmonic morphism because $\Delta_N y_1 = 0$ while $\Delta_M f_1 = \Delta_M(x_1 + x_2) = \frac{1}{2}(m-2)\exp x_1 \neq 0$.

Example 5.6. Let M denote $\mathbf{R}^3$ endowed with a Lorentz metric of the form

$$(g_M^{ij}) = \begin{pmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & a^{-2} \end{pmatrix}.$$

If $a(x) = \exp x_2$ the function $f(x) = (x_1)^2 \exp(-\frac{1}{2}x_2)$ of $x = (x_1, x_2, x_3)$ is harmonic on M:

$$\Delta_M f \equiv (2aD_1D_2 + D_2a \cdot D_1 + D_1a \cdot D_2 + a^{-2}D_3^2)f = 0.$$

The dilatation of f is

$$\lambda \equiv 2aD_1f \cdot D_2f + a^{-2}(D_3f)^2 = -2(x_1)^3,$$

which takes values > 0 and < 0; and f is non-degenerate because $\lambda(x) = 0$ only occurs when $\big(D_1f(x), D_2f(x), D_3f(x)\big) = (0,0,0)$.

If instead $a(x) = \exp(-x_1x_2)$, the analytic function

$$f(x) = \frac{1}{(x_2)^2}\big(\exp(x_1x_2) - 1 - x_1x_2\big) = (x_1)^2 \sum_{n=2}^{\infty} \frac{1}{n!}(x_1x_2)^{n-2}$$

is harmonic on M with the dilatation

$$\lambda = \frac{4}{(x_2)^4}\big(x_1x_2\sinh(x_1x_2) + 2 - 2\cosh(x_1x_2)\big) = 8(x_1)^4 \sum_{n=2}^{\infty} \frac{n-1}{(2n)!}(x_1x_2)^{2n-4}$$

taking the value 0 and values > 0, but not values < 0. Again, f is non-degenerate because $\lambda(x) = 0$ only occurs for $x_1 = 0$, where $\big(D_1f(x), D_2f(x), D_3f(x)\big) = (0,0,0)$. The map $f\colon M \to \mathbf{R}$ is *not open* because $f(x) \geqslant 0$ with equality when $x_1 = 0$.

Putting a minus sign in front of a in the definition of $g_M^{12} = g_M^{21}$ causes in either case λ to be replaced by $-\lambda$ ($\leqslant 0$ in the second case). Replacing $g_M^{33} = a^{-2}$ by $g_M^{33} = -a^{-2}$ makes $\operatorname{ind}_- M$ change from 1 to 2. Similar examples with $\dim M > 3$ and prescribed $\operatorname{ind}_- M$ ($\neq 0, \dim M$) are obtained by replacing the various (g_M^{ij}) described above with their direct sums with a suitable diagonal matrix with entries ± 1.

There is no 2-dimensional example serving the above purposes. In fact, for any 2-dimensional Lorentz manifold M the dilatation λ of any non-constant non-degenerate harmonic morphism of M into any semi-riemannian manifold N omits the value 0. (See Remark 3.2 for the case $\dim N = 2$; and use (8), Section 4, in the remaining case $\dim N = 1$, e.g. $N = R$, cf. Example 5.1 above.)

Appendix

This appendix is written by Lars Hörmander. His theorem below implies Lemma 3.2 when applied to the Laplace–Beltrami operator on N in local coordinates.

Theorem. *Let P be a second order differential operator in a neighborhood Ω of $0 \in \mathbf{R}^n$*

$$P = \sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^{n} b_i(x)\partial_i + c(x), \qquad \partial_i = \partial/\partial x_i,$$

with C^∞ coefficients, $a_{ij} = a_{ji}$ real valued and $\det a_{ij}(0) \neq 0$. Let m be an integer $\geqslant 2$. If $u \in C^\infty(\Omega)$ and $Pu(x) = O(|x|^{m-1})$ as $x \to 0$, then one can for any $s > m$ find $U \in C^s(\Omega)$ such that $U(x) - u(x) = O(|x|^{m+1})$ as $x \to 0$, and $PU = 0$ in a neighborhood of 0.

Proof. It is no restriction to assume that u is in the space $\mathscr{P}(m,n)$ of polynomials in $\mathbf{R}^n$ of degree $\leqslant m$ for we can replace u by its Taylor polynomial of degree m at the origin. Assuming as we may that $a_{nn}(0) \neq 0$, the polynomial u is then uniquely determined by the Cauchy data

$$u_0(x') = u(x',0), \qquad u_1(x') = \partial_n u(x',0), \qquad x' = (x_1,\dots,x_{n-1}),$$

which are in $\mathscr{P}(m,n-1)$ and $\mathscr{P}(m-1,n-1)$ respectively. In fact, the equations

$$(\partial_n^j Pu)(x',0) = O(|x'|^{m-1-j}) \qquad \text{as } x \to 0,\ 0 \leqslant j \leqslant m-2$$

are uniquely solvable for the derivatives of u at the origin of order $\leqslant m$. (This is the formal part of the Cauchy–Kovalevsky theorem.) Hence the dimension $d_{m,n}$ of the space of polynomials $u \in \mathscr{P}(m,n)$ such that $Pu(x) = O(|x|^{m-1})$ is equal to $\dim \mathscr{P}(m,n-1) + \dim \mathscr{P}(m-1,n-1)$ which is independent of P.

To prove the theorem we consider the equivalent operator

$$P_\varepsilon = \sum_{i,j=1}^{n} a_{ij}(\varepsilon x)\partial_i\partial_j + \varepsilon \sum_{i=1}^{n} b_i(\varepsilon x)\partial_i + \varepsilon^2 c(\varepsilon x), \qquad -1 \leqslant \varepsilon \leqslant 1,$$

obtained by a dilation when $\varepsilon > 0$. The statement of the theorem is trivial for the constant coefficient operator $P_0 = \sum a_{ij}(0)\partial_i\partial_j$ since $P_0 h(x) = O(|x|^{m-1})$ implies $P_0 h = 0$ if $h \in \mathscr{P}(m,n)$. Hence the space $H(m)$ of all $h \in \mathscr{P}(m,n)$ such that $P_0 h = 0$ has dimension $d_{m,n}$.

It suffices to prove the theorem for P_ε when $\varepsilon > 0$ is small. We may assume that Ω is a ball centered at the origin. Choose $\chi \in C_0^\infty(\Omega)$ equal to 1 in a neighborhood ω of 0. If $h \in H(m)$ then $(x,\varepsilon) \mapsto P_\varepsilon h(x)/\varepsilon$ is a C^∞ function since $P_0 h(x) = 0$. Set $f_\varepsilon^h(x) = \chi P_\varepsilon h(x)/\varepsilon$, which is bounded in $C_0^\infty(\Omega)$ as $\varepsilon \to 0$ for fixed h. For any positive integer s one can find $\delta_s > 0$ and a constant C_s such that for every f in the Sobolev space $H_{(s)}(\mathbf{R}^n)$ and every $\varepsilon \in (0, \delta_s)$ one can find $u \in H_{(s+1)}(\mathbf{R}^n)$ with $P_\varepsilon u = f$ in Ω depending linearly on f such that $\|u\|_{(s+1)} \leqslant C_s \|f\|_{(s)}$. If $s > \frac{1}{2}n + m$ it follows that

$$\sum_{|\alpha| \leqslant m+1} \sup_{\Omega} |\partial^\alpha u| \leqslant C_s' \|f\|_{(s)}.$$

In particular, we can find $u_\varepsilon^h \in C^{s+1-\frac{1}{2}n}$ such that $P_\varepsilon u_\varepsilon^h(x) = f_\varepsilon^h(x)$ in Ω, u_ε^h depends linearly on $h \in H(m)$, and $|\partial^\alpha u_\varepsilon^h(0)| \leqslant C|h|$, $|\alpha| \leqslant m$, where $|\cdot|$ denotes a norm in the finite dimensional vector space $H(m)$. Hence $U_\varepsilon^h(x) = h(x) - \varepsilon u_\varepsilon^h(x)$ satisfies the equation $P_\varepsilon U_\varepsilon^h = 0$ in ω, and the Taylor polynomial of U_ε^h of order m is $h - \varepsilon T_\varepsilon h$ where $T_\varepsilon \colon H(m) \to \mathscr{P}(m,n)$ is linear and has a bound independent of ε as $\varepsilon \to 0$. The rank of the map $H(m) \ni h \mapsto h - \varepsilon T_\varepsilon h \in \mathscr{P}(m,n)$ is therefore equal to $\dim H(m) = d_{m,n}$ for small ε. The range is contained in the space $H_\varepsilon(m)$ of polynomials $u \in \mathscr{P}(m,n)$ such that $P_\varepsilon u(x) = O(|x|^{m-1})$. Since $\dim H_\varepsilon(m) = d_{m,n}$ it follows that the range is equal to $H_\varepsilon(m)$, which proves the theorem. □

Remark. In the proof we have only used a very simple existence theorem for operators of real principal type, which can be found already in Hörmander [7]. The argument above based on this reference is valid with no essential change for operators of higher order. For the second order case one could also derive the required existence theorem from the Hadamard parametrix construction. Using more sophisticated existence theorems which can be found in [9, Chap. XXVI], one can obtain $U \in C^\infty$ in the theorem.

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