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THE CHORDAL NORM OF DISCRETE MÖBIUS GROUPS IN SEVERAL DIMENSIONS

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Abstract. Let $d(f,g)=\sup\{d(f(z),g(z)) : z\in\overline{\mathbf{C}}\}$ where f, g are Möbius transformations and $d(z_1,z_2)$ denotes the chordal distance between z_1, z_2 in $\overline{\mathbf{C}}$. We show that if $\langle f,g\rangle$ is a discrete group and if $fg\neq gf$, then

$$\max\{d(f,\mathrm{id}),d(g,\mathrm{id})\}\geq c$$

where $.863\leq c\leq .911\cdots$. We also obtain some higher dimensional analogs by means of Clifford numbers.

1. Introduction

Let $\mathrm{GM}(n)$ denote the group of all Möbius transformations of $\overline{\mathbf{R}}^n$ and $\mathrm{M}(n)$ the subgroup of $\mathrm{GM}(n)$ consisting of all orientation-preserving Möbius transformations. Stereographic projection p is the mapping from $\overline{\mathbf{R}}^n$ onto the unit sphere S^n in $\mathbf{R}^{n+1}$ given by

$$p(x)=e_{n+1}+\frac{2(x-e_{n+1})}{|x-e_{n+1}|^2}$$

where e_1, $e_2,\dots,e_n$ is the standard basis for $\overline{\mathbf{R}}^n$. The chordal distance between two points x and y in $\overline{\mathbf{R}}^n$ is defined by

$$d(x,y)=|p(x)-p(y)|.$$

The chordal metric on $\mathrm{GM}(n)$ is set to be

$$d(f,g)=\sup\bigl\{d\bigl(f(x),g(x)\bigr) : x\in\overline{\mathbf{R}}^n\bigr\}.$$

This metric was considered in [3] and [9]. We call $d(f)=d(f,\mathrm{id})$ the chordal norm of f. Then $d(f)$ measures the maximum chordal derivation of f from the identity, $0\leq d(f)\leq 2$ and $d(f)=2$ if and only if f maps one point of a pair of antipodal points of $\overline{\mathbf{R}}^n$ onto the other.

A subgroup G of $\mathrm{M}(n)$ is discrete if there exists a positive constant $k=k(G)$ such that $d(f,g)\geq k$ for each distinct f and g in G. Martin shows that a dimension dependent lower bound can be obtained for chordal norms [14].

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1.1. Theorem (Martin). *Let* f *and* g *be two Möbius transformations of* $\mathbf{B}^n$ *generating a discrete subgroup. Then*

$$\max\{d(f), d(g)\} \geq \frac{1}{2\sqrt{16+n}}$$

unless $\langle f, g\rangle$ *is an elementary nilpotent group.*

For plane Möbius transformations, Gehring and Martin [9] obtained

1.2. Theorem (Gehring–Martin). *Suppose that* $\langle f, g\rangle$ *is a nonelementary discrete subgroup of* $\mathbf{M}$. *Then*

$$\max\{d(f), d(g)\} \geq a$$

where $2(\sqrt{2}-1) = 0.828\cdots \leq a \leq 0.911\cdots$.

By means of Clifford numbers, we show in Section 3 that if f and g generate a discrete nonelementary subgroup of $\mathrm{M}(n)$, then

$$\begin{aligned} &\max\{d(f), d(g)\} \geq .683, && \text{if } f \text{ is hyperbolic,}\\ &\max\{d(f), d(g)\} \geq 1.22, && \text{if } f \text{ is strictly parabolic,}\\ &\max\{d(f), d(g)\} \geq .816, && \text{if } f \text{ is loxodromic and fixes } 0 \text{ and } \infty. \end{aligned}$$

For plane Möbius transformations, we show that if $\langle f, g\rangle$ is a discrete subgroup of Möbius transformations of $\overline{\mathbf{C}}$ and if $fg \neq gf$, then

$$\begin{aligned} &\max\{d(f), d(g)\} \geq c_1, && \text{if } f \text{ is parabolic,}\\ &\max\{d(f), d(g)\} \geq c_2, && \text{if } f \text{ is elliptic,}\\ &\max\{d(f), d(g)\} \geq c_3, && \text{if } f \text{ is loxodromic,} \end{aligned}$$

where $c_1 = 1$, $.937 \leq c_2 \leq 1.12\cdots$, $.863 \leq c_3 \leq .911\cdots$. We then apply these results to get some necessary and sufficient conditions for a group to be discrete.

2. Clifford numbers

In [1] Ahlfors shows how a 2×2 matrix with entries in a Clifford algebra may be used to describe a Möbius transformation of $\overline{\mathbf{R}}^n$. In this section we will briefly review some material on Clifford numbers that is treated in detail in [1] and [2].

The Clifford algebra C_n is the associative algebra over the reals generated by elements i_1, $i_2, \ldots, i_n$ subject to the relations $i_k^2 = -1$ and $i_h i_k = -i_k i_h$, $h \neq k$, and no others. An element of C_n is called a Clifford number. Every Clifford number a can be expressed uniquely in the form $a = \sum a_I I$ where $a_I \in \mathbf{R}$ and the sum is over all products $I = i_{\nu_1} i_{\nu_2} \cdots i_{\nu_p}$ with $1 \leq i_{\nu_1} < \cdots < i_{\nu_p} \leq n$. The null product is included and identified with the real number $i_0 = 1$.

The coefficient of the empty product is denoted by a_0 and called the real part of a. The sum of all other terms of a is referred to as the imaginary part and we write $a = \mathrm{Re}(a) + \mathrm{Im}(a)$. We sometimes denote $\mathrm{Im}(a)$ by a_c. The Euclidean norm of a Clifford number is given by $|a|^2 = \sum a_I{}^2 = \mathrm{Re}(a)^2 + |a_c|^2$.

There are three involutions of C_n. The major involution $a \to a'$ replaces each i_k by $-i_k$. It determines an automorphism of C_n: $(ab)' = a'b'$, $(a+b)' = a' + b'$. The reversion $a \to a^*$ replaces each $I = i_{\nu_1} i_{\nu_2} \cdots i_{\nu_p}$ with $I = i_{\nu_p} \cdots i_{\nu_2} i_{\nu_1}$. It defines an anti-automorphism: $(ab)^* = b^* a^*$, $(a+b)^* = a^* + b^*$. The third involution $a \to \bar{a}$ is a composition: $\bar{a} = a'^* = a^{*\prime}$, which is again an anti-automorphism.

We identify $\mathbf{R}^n$ with the subspace spanned by 1, i_1, $i_2, \ldots, i_{n-1}$. Clifford numbers of the form $x = x_0 + x_1 i_1 + \cdots + x_{n-1} i_{n-1}$ are called vectors. Every non-zero vector x is invertible with $x^{-1} = \bar{x}|x|^{-2}$. The product of nonzero vectors form a multiplicative group Γ_n, known as *Clifford group*.

A Clifford matrix of dimension n is a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which satisfies the conditions

(1) $a,\ b,\ c,\ d \in \Gamma_n \cup \{0\}$,

(2) $ad^* - bc^* = 1$,

(3) $ab^*,\ cd^*,\ c^*a,\ d^*b \in \mathbf{R}^n$.

The set of all Clifford matrices is denoted by $\mathrm{SL}(2, C_n)$. It is Vahlen's theorem that $\mathrm{SL}(2, C_n)$ form a group whose quotient modulo $\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is isomorphic to $\mathrm{M}(\overline{\mathbf{R}}^n)$, the group of orientation preserving transformations of $\overline{\mathbf{R}}^n$ [2]. A Clifford matrix in dimension n is also a Clifford matrix in dimension $n+1$. It automatically extends the corresponding transformation in $\mathrm{M}(\overline{\mathbf{R}}^n)$ to one in $\mathrm{M}(\overline{\mathbf{R}}^{n+1})$ given by the same matrix.

The following is the classification of Möbius transformations.

$f \in \mathrm{GM}(n)$ is elliptic if it has a fixed point in $\mathbf{H}^{n+1}$. Such maps are $\mathrm{GM}(n+1)$ conjugate to $x \to Tx$ with $T \in \mathrm{O}(n)$.

$f \in \mathrm{GM}(n)$ is parabolic if it has exactly one fixed point, necessarily in $\overline{\mathbf{R}}^n$. Such maps are $\mathrm{GM}(n)$ conjugate to $x \to Tx + a$ with $T \in \mathrm{O}(n)$, $a \in \mathbf{R}^n \setminus \{0\}$. A parabolic which is conjugate to $x \to x + a$ is called strictly parabolic.

$f \in \mathrm{GM}(n)$ is loxodromic if it has exactly two fixed points, both in $\overline{\mathbf{R}}^n$. Such maps are $\mathrm{GM}(n)$ conjugate to $x \to rTx$ with $r > 0$, $r \neq 1$ and $T \in \mathrm{O}(n)$. If $T = I$, then f is called hyperbolic.

A subgroup G of $\mathrm{GM}(n)$ is said to be elementary if and only if there exists a finite G-orbit in $\mathbf{R}^{n+1}$.

3. Möbius transformations of $\overline{\mathbf{R}}^n$

3.1. Lemma. *Suppose that f is not elliptic. Define the map θ: $\mathrm{GM}(n) \to$*

GM(n) by $\theta(g) = gfg^{-1}$. If for some n, $\theta^n(g)$ and f have the same fixed point set, then $g(\mathrm{fix}(f)) = \mathrm{fix}(f)$ and hence $\langle f, g\rangle$ is elementary.

Proof. The result follows from the corresponding statement for $\mathrm{SL}(2, \mathbf{C})$ [3, Theorem 5.1.4] and the fact that a parabolic or loxodromic element has at most two fixed points in $\overline{\mathbf{R}}^n$. □

3.2. Lemma. *Suppose that Möbius transformations f and g are represented by*

$$A = \begin{pmatrix} u & 0 \\ 0 & {u^*}^{-1} \end{pmatrix}, \qquad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad^* - bc^* = 1. \tag{3.3}$$

Suppose that f is loxodromic and that g does not keep the fixed point set of f invariant. If $\langle f, g\rangle$ is discrete, then

$$d(f)^2 \geq 4(|u| - |u|^{-1})^2/(|u| + |u|^{-1})^2, \tag{3.4}$$

$$d(g)^2 \geq 4|bc|/(1 + 2|bc|), \tag{3.5}$$

$$d(f)^2(1 + |bc|) \geq 4/(|u| + |u|^{-1})^2. \tag{3.6}$$

Proof. Let $x = |u|^{-1}$. Then

$$d^2(f(x), x) \geq \frac{4(|f(x)| - |x|)^2}{(1 + |f(x)|^2)(1 + |x|^2)} = 4(|u| - |u|^{-1})^2/(|u| + |u|^{-1})^2.$$

This proves (3.4). For (3.5), we let $x_1 = 0$, $x_2 = \infty$. Since

$$d(g)^2 \geq d^2(g(x_1), x_1) = \frac{4|bd^{-1}|^2}{1 + |bd^{-1}|^2},$$

$$d(g)^2 \geq d^2(g(x_2), x_2) = \frac{4}{1 + |ac^{-1}|^2},$$

we have $(4 - d(g)^2)|bc| \leq d(g)^2|ad|$. So (3.5) follows.

The proof of (3.6) requires the discreteness of $\langle f, g\rangle$. It is essentially due to Hersonsky [11], Friedland and Hersonsky [6], and Waterman [15]. We include a proof for completeness. Consider the Shimizu–Leutbecher sequence

$$B_0 = B, \qquad B_{n+1} = B_n A {B_n}^{-1}. \tag{3.7}$$

The relation (3.7) yields

$$\begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix} = \begin{pmatrix} a_n u d_n^* - b_n {u^*}^{-1} c_n^* & -a_n u b_n^* + b_n {u^*}^{-1} a_n^* \\ c_n u d_n^* - d_n {u^*}^{-1} c_n^* & -c_n u b_n^* + d_n {u^*}^{-1} a_n^* \end{pmatrix}. \tag{3.8}$$

We observe that if one of a_n, b_n, c_n, d_n is zero, then $b_{n+1}c_{n+1} = 0$. For each n with $a_n b_n c_n d_n \neq 0$, we let

$$x_n = a_n^{-1} b_n, \qquad y_n = c_n^{-1} d_n, \qquad p_n = x_n/|ux_n|, \qquad q_n = y_n/|uy_n|.$$

Then

$$\begin{aligned} |b_{n+1}c_{n+1}| &= |a_n b_n c_n d_n||u - x_n u^{*-1} x_n^{-1}||u - y_n u^{*-1} y_n^{-1}| \\ &= |a_n b_n c_n d_n| d\big(f(p_n), p_n\big) d\big(f(q_n), q_n\big)(|u| + |u|^{-1})^2/4 \\ &\leq |b_n c_n|(1 + |b_n c_n|) d(f)^2 (|u| + |u|^{-1})^2/4. \end{aligned} \tag{3.9}$$

Suppose that $\mu = (1 + |bc|) d(f)^2 (|u| + |u|^{-1})^2/4 < 1$. We will obtain a contradiction. We obtain, by induction,

$$|b_n c_n| \leq \mu^n |bc| \leq |bc|.$$

So,

$$b_n c_n \to 0, \qquad a_n d_n^* \to 1.$$

It follows from (3.8) that

$$|a_n| \to |u|, \qquad |d_n| \to |u|^{-1}.$$

Now

$$|b_{n+1}|/|b_n| \leq |a_n| d(f)(|u| + |u|^{-1})/2.$$

Thus, by induction, $|b_n|/|u|^n \to 0$, and similarly, $|c_n||u|^n \to 0$. So

$$A^{-n} B_{2n} A^n = \begin{pmatrix} u^{-n} a_{2n} u^n & u^{-n} b_{2n} u^{*-n} \\ u^{*n} c_{2n} u^n & u^{*n} d_{2n} u^{*-n} \end{pmatrix}$$

has a subsequence that converges to a diagonal matrix. Since $\langle f, g\rangle$ is discrete, $u^{-n_k} b_{2n_k} u^{*-n_k} = 0$ and $u^{*n_k} c_{2n_k} u^{n_k} = 0$ for sufficiently large k. Thus $b_n = c_n = 0$ for infinitely many n. Hence $g\big(\mathrm{fix}(f)\big) = \mathrm{fix}(f)$ by Lemma 3.1. This contradicts the assumption that g does not keep the fixed point set of f invariant. Therefore $\mu \geq 1$. □

3.10. Corollary. *Suppose that f is loxodromic with $\mathrm{fix}(f) = \{0, \infty\}$ and that g does not keep the fixed point set of f invariant. If $\langle f, g\rangle$ is a discrete subgroup of $\mathrm{M}(n)$, then*

$$\max\{d(f), d(g)\} \geq .816.$$

Proof. Since $\mathrm{fix}(f) = \{0, \infty\}$, f and g can be represented by the Clifford matrices A and B as in (3.3). Let $k = .816$, $s = 1.543$. If $|u| \geq s$ or $|u|^{-1} \geq s$, then $d(f) \geq k$ by (3.4). If $|bc| \geq \frac{1}{4}$, then $d(g) \geq k$ by (3.5). Finally, if $1/s < |u| < s$ and $|bc| < \frac{1}{4}$, then (3.6) implies that $d(f) \geq k$.

3.11. Theorem. *Suppose that $\langle f, g\rangle$ is a discrete subgroup of $\mathrm{M}(n)$. If f is hyperbolic and g does not keep the fixed point set of f invariant, then*

$$\max\{d(f), d(g)\} \geq .683.$$

Proof. The statement is invariant with respect to conjugation by chordal isometries. Thus by means of such a conjugation we may assume that $f(\infty)=\infty$. Thus f and g can be represented by

$$A=\begin{pmatrix} u & t \\ 0 & u^{*-1}\end{pmatrix}, \qquad B=\begin{pmatrix} a & b \\ c & d\end{pmatrix}, \qquad ad^*-bc^*=1$$

where u is real. Replacing f by f^{-1} if necessary, we may assume $|u|>1$. Let w be the other fixed point of f. Then w satisfies the equation

$$uw+t=wu^{*-1}. \tag{3.12}$$

Let h be the Möbius transformation represented by $C=\begin{pmatrix} 1 & -w \\ 0 & 1\end{pmatrix}$. Then

$$CAC^{-1}=\begin{pmatrix} u & 0 \\ 0 & u^{*-1}\end{pmatrix}, \qquad CBC^{-1}=\begin{pmatrix} a-wc & aw+b-w(cw+d) \\ c & cw+d\end{pmatrix}.$$

We now consider the sequence

$$B_{n+1}=B_nCAC^{-1}B_n^{-1}, \qquad B_0=CBC^{-1}.$$

We obtain as in the proof of Lemma 3.2 (see (3.9)) that

$$|b_{n+1}c_{n+1}|=|a_nb_nc_nd_n|\,|\beta|$$

where $\beta=(u-u^{*-1})^2$. The same argument as in the proof of Lemma 3.2 yields

$$|\beta|+|\beta\big(aw+b-w(cw+d)\big)c|\geq 1. \tag{3.13}$$

Since

$$\begin{aligned} |aw+b|^2+|cw+d|^2 &\leq \|g\|^2(1+|w|^2),\\ |t|^2\big(4-d^2\big(f(0),0\big)\big) &\leq d^2(f(0),0), \qquad (|u|>1)\\ 4|c|^2 &\leq \|g\|^2d^2\big(g(\infty),\infty\big), \end{aligned}$$

it follows from (3.13) and (3.12) that

$$|\beta|+\frac{1}{4}\Big(|\beta|+\frac{d(f)^2}{4-d(f)^2}\Big)\|g\|^2d(g)^2\geq 1. \tag{3.14}$$

Now suppose that

$$\max\{d(f), d(g)\} < .683. \tag{3.15}$$

We will obtain a contradiction. It is a consequence of [7, Theorem 3.3] that

$$\|g\|^2 \le 2\frac{4+d(g)^2}{4-d(g)^2}.$$

Hence if $|\beta| \le .742$, then either $d(f) \ge .683$ or $d(g) \ge .683$ by (3.14). This contradicts (3.15).

Suppose that $|\beta| > .742$. Notice that

$$4d(f)^{-2} = \inf_{x\in\overline{\mathbf{R}}^n} \frac{|f(x)\bar{x}+1|^2}{|f(x)-x|^2} + 1.$$

Let $x = t/|ut|$. We obtain

$$4d(f)^{-2} - 1 \le (|t|+2)^2/|\beta|. \tag{3.16}$$

Since $|t|^2 \le d(f)^2/\big(4-d(f)^2\big)$, (3.16) yields $d(f) > .683$. This contradicts (3.15). ◻

3.17. Theorem. *Suppose that* $\langle f, g\rangle$ *is a discrete subgroup of* $\mathrm{M}(n)$. *If* f *is strictly parabolic and* g *does not fix the fixed point of* f, *then*

$$\max\{d(f), d(g)\} \ge 1.22.$$

Proof. The statement is invariant with respect to conjugation by chordal isometries. Thus by means of such a conjugation we may assume that $f(\infty) = \infty$. Hence f and g can be represented by

$$A = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad^* - bc^* = 1.$$

By [15, Lemma 2.1], $|tc| \ge 1$. Since

$$\begin{aligned} d^2\big(f(0), 0\big) &= |t|^2\big(4 - d^2\big(f(0),0\big)\big), \\ 4|c|^2 &\le \|g\|^2 d^2\big(g(\infty), \infty\big), \\ \|g\|^2 &\le 2\frac{4+d(g)^2}{4-d(g)^2}, \qquad \text{[7, Theorem3.1]} \end{aligned}$$

we have

$$\frac{d(f)^2 d(g)^2\big(4+d(g)^2\big)}{\big(4-d(f)^2\big)\big(4-d(g)^2\big)} \ge 2.$$

Therefore $\max\{d(f), d(g)\} \ge 1.22$. ◻

4. Plane Möbius transformations

Let $\mathbf{M}$ denote the group of all orientation-preserving Möbius transformations of the extended complex plane $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. We associate with each

$$f = \frac{az+b}{cz+d} \in \mathbf{M}, \qquad ad - bc = 1,$$

the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{C})$$

and set $\mathrm{tr}(f) = \mathrm{tr}(A)$, where $\mathrm{tr}(A)$ denotes the trace of A. Note that $\mathrm{tr}(f)$ is defined up to sign. The matrix norm $m(f)$ is defined by (see [9])

$$m(f) = \|A - A^{-1}\| = (2|a-d|^2 + 4|b|^2 + 4|c|^2)^{1/2}.$$

The quantity $\|A - A^{-1}\|$ is independent of the choice of A representing f and is invariant with respect to conjugation by chordal isometries.

For each f and g in $\mathbf{M}$ we let $[f,g]$ denote the multiplicative commutator $fgf^{-1}g^{-1}$. We call the three complex numbers

$$\beta(f) = \mathrm{tr}^2(f) - 4, \qquad \beta(g) = \mathrm{tr}^2(g) - 4, \qquad \gamma(f,g) = \mathrm{tr}([f,g]) - 2,$$

the parameters of the two generator group $\langle f, g\rangle$. These parameters are independent of the choice of representative matrices for f and g, and they determine $\langle f, g\rangle$ up to conjugacy whenever $\gamma(f,g) \neq 0$ [8]. But see [4] for three generator Möbius groups. Note that $\gamma(f,g) \neq 0$ if and only if f and g do not have a common fixed point in $\overline{\mathbf{C}}$.

There are some necessary conditions for a two generator group to be discrete.

4.1. Theorem. *If $\langle f, g\rangle$ is discrete with $\gamma(f,g) \neq 0$ and $\gamma(f,g) \neq \beta(f)$, then*

$$|\gamma(f,g)| + |\beta(f)| \geq 1.$$

4.2. Theorem. *If $\langle f, g\rangle$ is discrete with $\gamma(f,g) \neq 0$ and $\gamma(f,g) \neq \beta(f)$ and if $|\beta(f)| \leq 2\big(\cos(2\pi/7) + \cos(\pi/7) - 1\big) = 1.0489\ldots$, then*

$$|\gamma(f,g)| \geq 2 - 2\cos(\pi/7) = 0.198\ldots .$$

Theorem 4.1 is due to Jørgensen [12]. A proof of Theorem 4.2 is given in [5]. We will quantify the above statements in terms of matrix and chordal norms. See [9] for related results.

4.3. Lemma. *Suppose that $\langle f,g\rangle$ is a discrete subgroup of* $\mathbf{M}$ *with* $\gamma(f,g)\neq 0$ *and* $\gamma(f,g)\neq\beta(f)$. *If* f *is loxodromic, then*

$$m(f)m(g)\geq 4\bigl(2-2\cos(\pi/7)\bigr)^{1/2}=1.78\ldots. \tag{4.4}$$

If f *is elliptic of order greater than two, then*

$$m(f)m(g)\geq 4\bigl(2\cos(2\pi/7)-1\bigr)^{1/2}=1.987\ldots. \tag{4.5}$$

Inequality (4.5) *is sharp.*

Proof. Let $c_0=2\bigl(\cos(2\pi/7)+\cos(\pi/7)-1\bigr)$, $d_0=2-2\cos(\pi/7)$. Suppose that f is loxodromic. If $|\beta(f)|\geq c_0$ and $|\beta(g)|\geq c_0$, then

$$m(f)m(g)\geq 2|\beta(f)\beta(g)|^{1/2}\geq 2c_0=2.097\ldots$$

by [9, Theorem 2.7]. If $|\beta(g)|\leq c_0$, then $|\gamma(f,g)|\geq d_0$ by [5, Theorem 3.1]. If $|\beta(f)|\leq c_0$, then $|\gamma(f,g)|\geq d_0$ by Theorem 4.2. Thus by [9, Theorem 2.7],

$$m(f)m(g)\geq 4|\gamma(f,g)|^{1/2}\geq 4\sqrt{d_0}=1.78\ldots.$$

Suppose next that f is elliptic of order greater than two. Then by [10, Theorem 3.1], $|\gamma(f,g)|\geq 2\cos(2\pi/7)-1$. Therefore,

$$m(f)m(g)\geq 4\bigl(2\cos(2\pi/7)-1\bigr)^{1/2}.$$

The $(2,3,7)$ triangle group in [9, Lemma 4.8] shows that (4.5) is sharp. □

4.6. Remark. (i) Gehring and Martin [9] have shown that if $\langle f,g\rangle$ is nonelementary discrete, then

$$m(f)m(g)\geq 4(\sqrt{2}-1)=1.656\ldots.$$

Furthermore, $m(f)m(g)\geq 4$ if f is parabolic.

(ii) Let f, g be the Möbius transformations represented by

$$A=\begin{pmatrix}\cos(\pi/n) & i\sin(\pi/n)\\ i\sin(\pi/n) & \cos(\pi/n)\end{pmatrix},\qquad B=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}.$$

Then $\langle f,g\rangle$ is discrete and $\gamma(f,g)=\beta(f)=-4\sin^2(\pi/n)$. In this case,

$$m(f)m(g)=8\sin(\pi/n)\to 0\qquad\text{as}\quad n\to\infty.$$

Notice that $\gamma(g,f)\neq\beta(g)$ and g is elliptic of order two.

(iii) If f is of order two and $\gamma(f,g)\neq 0$, $\gamma(f,g)\neq\beta(g)$, then $m(f)m(g)\geq 2\sqrt{2}$. This is because that if $|\beta(g)|\geq 1/2$, then $m(f)m(g)\geq 2|\beta(f)\beta(g)|^{1/2}\geq\sqrt{8}$. If $|\beta(g)|\leq 1/2$, then $|\gamma(f,g)|\geq 1/2$ by Jørgensen's inequality. So $m(f)m(g)\geq 4|\gamma(f,g)|^{1/2}\geq\sqrt{8}$.

4.7. Lemma. *Let $e(c)=\inf\{d(f): f \text{is elliptic and } m(f)=c\}$. Then*

$$e(c)=\begin{cases}\dfrac{c}{\sqrt{2}} & \text{if } 0<c\le 4(\sqrt{2}-1)^{1/2},\\ \dfrac{16c}{c^2+16} & \text{if } 4(\sqrt{2}-1)^{1/2}<c<4,\\ 2 & \text{if } 4\le c.\end{cases} \tag{4.8}$$

Proof. All quantities in (4.8) are invariant with respect to conjugation by chordal isometries. Thus by means of such a conjugation we may arrange that $\operatorname{fix}(f)=\{-r,r\}$ where $0<r\le 1$. Since f is elliptic, it is conjugate to a mapping of the form $w=e^{i2\theta}z$, $-\pi/2<\theta\le\pi/2$. By [9, Lemma 2.10],

$$\begin{aligned} m(f)^2 &= 2\big(8-q(-r,r)^2\big)q(-r,r)^{-2}|\beta(f)|\\ &= (r^2+r^{-2})|e^{i\theta}-e^{-i\theta}|^2\\ &= (r^2+r^{-2})4\sin^2\theta. \end{aligned}$$

Note that $m(f)\ge 4$ if and only if $|\tan(\theta/2)|\ge r$. Thus by [9, Lemma 3.1], $d(f)=2$ if $c\ge 4$. We now assume that $|\tan(\theta/2)|<r$. Then [9, Lemma 3.1] implies that

$$16d(f)^{-2}-2=r^2\cot^2(\theta/2)+r^{-2}\tan^2(\theta/2).$$

We will find the maximum value of

$$g(r,\theta)=r^2\cot^2(\theta/2)+r^{-2}\tan^2(\theta/2),\qquad 0<r\le 1,$$

subject to the constraint

$$h(r,\theta)=4(r^2+r^{-2})\sin^2\theta-c^2=0.$$

Let $F(r,\theta)=g(r,\theta)+\lambda h(r,\theta)$. Then the critical points satisfy the equations:

$$\frac{\partial F}{\partial r}=0,\qquad \frac{\partial F}{\partial\theta}=0.$$

It follows that

$$r^2\cot^2(\theta/2)-r^{-2}\tan^2(\theta/2)=4\lambda(r^{-2}-r^2)\sin^2\theta, \tag{4.9}$$

$$r^2\cot^2(\theta/2)-r^{-2}\tan^2(\theta/2)=4\lambda(r^{-2}+r^2)\sin^2\theta\cos\theta. \tag{4.10}$$

Since $|\tan(\theta/2)|<r$, (4.9)/(4.10) gives $\cos\theta=(1-r^4)(1+r^4)^{-1}$. Hence at the critical points,

$$g(r,\theta)=16/c^2. \tag{4.11}$$

Next we consider the case $r\to 0$. Solving θ from the equation $h(r,\theta)=0$, we get

$$\lim_{r\to 0} g(r,\theta)=\lim_{r\to 0}\big(r^2\cot^2(\theta/2)+r^{-2}\tan^2(\theta/2)\big)=16/c^2+c^2/16. \tag{4.12}$$

At the end point $r=1$, $\sin^2\theta=c^2/8$. Hence

$$g(1,\theta)=32/c^2-2. \tag{4.13}$$

Combining (4.11), (4.12) and (4.13), we obtain (4.8). □

4.14. Remark. Let $p(c) = \inf\{d(f) : f$ is parabolic and $m(f) = c\}$ and let $l(c) = \inf\{d(f) : f$ is loxodromic and $m(f) = c\}$. Then

$$l(c) = \frac{2c}{(c^2+8)^{1/2}}, \qquad p(c) = \begin{cases} \dfrac{16c}{c^2+16} & \text{if } 0 < c < 4, \\ 2 & \text{if } 4 \le c \end{cases}$$

by [9, Theorem 3.11 and Lemma 3.8]. It is easy to check that $p(c)$, $e(c)$, $l(c)$ are continuous increasing functions of c and $p(c) \ge e(c) > l(c)$.

4.15. Lemma. *Suppose that $\langle f, g\rangle$ is a discrete subgroup of $\mathbf{M}$ and that f is elliptic. If $fg \ne gf$, then*

$$\max\{d(f), d(g)\} \ge c_1, \qquad .937 \le c_1 \le 1.121\dots .$$

Proof. Suppose first that $\gamma(f,g) \ne 0$ and $\gamma(f,g) \ne \beta(f)$. If f is of order two, then $d(f) = 2$ by [9, Corollary 3.17]. Suppose that f is elliptic of order greater than two. Let $a_0 = 4\bigl(2\cos(2\pi/7) - 1\bigr)^{1/2}$, $t = m(f)$. Then $m(g) \ge a_0/t$ by Lemma 4.3. By Remark 4.14,

$$\max\{d(f), d(g)\} \ge \max\bigl\{e\bigl(m(f)\bigr), l\bigl(m(g)\bigr)\bigr\} \ge \max\{e(t), l(a_0/t)\}.$$

Note that $e(x)$ is strictly increasing for $x \le 4$ and $l(a_0/x)$ is strictly decreasing. Hence $\max\{e(x), l(a_0/x)\}$ obtains its minimum when x is the intersection point x_0 of $e(x)$ and $l(a_0/x)$. Solving for x, we get

$$x_0 = \bigl((a_0^2 + (a_0/4)^4)^{1/2} - (a_0/4)^2\bigr)^{1/2},$$
$$\max_{x>0}\{e(x), l(a_0/x)\} \ge x_0/\sqrt{2} = .937\dots .$$

Suppose next that $\gamma(f,g) \ne 0$, $\gamma(f,g) = \beta(f)$. Then either f is elliptic of order 2, 3, 4, or 6 or g is elliptic of order 2 by [10, Lemma 2.31]. In either case,

$$\max\{d(f), d(g)\} \ge 2\sin(\pi/6) = 1$$

by [9, Corollary 3.17]. Finally, suppose that $\gamma(f,g) = 0$. Then f and g have a common fixed point in $\overline{\mathbf{C}}$, say ∞. Thus every element of $\langle f, g\rangle$ fixes ∞. Since $fg \ne gf$, $[f,g]$ is parabolic by [3, Theorem 4.3.5]. So there are no loxodromic elements in $\langle f, g\rangle$ by [3, Theorem 5.1.2]. By the structure of elementary groups [3, § 5.1], if S is the set of multipliers of $\langle f, g\rangle$, then $\mathrm{S} = \{1, \omega, \omega^2, \dots, \omega^{q-1}\}$ where $\omega = \exp(2\pi i/q)$, $0 \le q \le 6$, $q \ne 5$. So f is elliptic of order less than or equal to six. Therefore,

$$\max\{d(f), d(g)\} \ge d(f) \ge 2\sin(\pi/6) = 1.$$

To get the number $1.121\ldots$, let $\langle\phi,\psi\rangle$ denote the $(2,3,7)$ triangle group with $\phi^2=\psi^3=(\phi\psi)^7=\mathrm{id}$. The transformations ϕ and ψ can be represented by the matrices

$$A=\frac{i}{\sin a}\begin{pmatrix}-\cos b & -p\\ p & \cos b\end{pmatrix},\qquad B=\begin{pmatrix}e^{ia} & 0\\ 0 & e^{-ia}\end{pmatrix}$$

where $a=\pi/3$, $b=\pi/7$ and $p=(\cos^2 b-\sin^2 a)^{1/2}$ [13, p. 88]. We set $f=[A,B]$ and $g=AB$. Then

$$\beta(f)=2\bigl(\cos(2\pi/7)+\cos(\pi/7)-1\bigr),\qquad \beta(g)=2\cos(2\pi/7)-2,$$
$$\gamma(f,g)=2\cos(2\pi/7)-1.$$

We can find a Möbius transformation h which sends the fixed points of f to $\{w,-w\}$ and sends the fixed points of g to $\{1/w,-1/w\}$. By [9, Lemma 2.12], such a w satisfies the equation

$$(w^2-1/w^2)^2=16\frac{\gamma(f,g)}{\beta(f)\beta(g)}.$$

Let $u=|w|^2+1/|w|^2$. Then $m^2(hfh^{-1})=u\beta(f)$ and $m^2(hgh^{-1})=-u\beta(g)$ by [9, Lemma 2.10]. It is a consequence of [9, Lemma 3.1] that for any Möbius transformation f, if $m(f)^2\le 2(|\beta+4|+4)$, then

$$\begin{aligned}d(f)=&\frac{2(|\beta+4|+4+|\beta|)\bigl(\frac12 m(f)^2+|\beta|)\bigr)^{1/2}}{|\beta+4|+4+\frac12 m(f)^2}\\&+\frac{2(|\beta+4|+4-|\beta|)\bigl(\frac12 m(f)^2-|\beta|)\bigr)^{1/2}}{|\beta+4|+4+\frac12 m(f)^2}.\end{aligned}$$

Therefore

$$d(hfh^{-1})=1.121\ldots,\qquad d(hgh^{-1})=1.071\ldots.\ \square$$

4.16. Example. Let $f=e^{2\pi i/m}z$, $g=e^{2\pi i/n}z$. Then $\langle f,g\rangle$ is a discrete finite group with $fg=gf$. We have $d(f)\to 0$, $d(g)\to 0$, as $m,n\to\infty$.

4.17. Lemma. *Suppose that $\langle f,g\rangle$ is a discrete subgroup of* $\mathbf{M}$ *and that f is parabolic. If $fg\ne gf$, then*

$$\max\{d(f),d(g)\}\ge 1.$$

Proof. If $\gamma(f,g)\ne 0$, then $m(f)m(g)\ge 4$ by [9, Lemma 4.5]. Let $t=m(f)$. By Remark 4.14,

$$\max\{d(f),d(g)\}\ge\max\bigl\{p\bigl(m(f)\bigr),l\bigl(m(g)\bigr)\bigr\}\ge\max\{p(t),l(4/t)\}.$$

Note that $p(x)$ is strictly increasing for $x \le 4$ and $l(4/x)$ is strictly decreasing. Hence $\max\{p(x), l(4/x)\}$ obtains its minimum when x is the intersection point x_0 of $p(x)$ and $l(4/x)$. Solving for x_0, we have

$$x_0 = \frac{4}{31}(124\sqrt{2} - 31)^{1/2},$$
$$\max_{x>0}\{p(x), l(4/x)\} \ge 4/(4 + 2x_0^2)^{1/2} = 1.347\dots .$$

So,

$$\max\{d(f), d(g)\} \ge 1.347\dots .$$

If $\gamma(f,g) = 0$, then f and g have a common fixed point in $\overline{\mathbf{C}}$, say ∞. Thus every element of $\langle f, g\rangle$ fixes ∞. Since f is parabolic, there are no loxodromic elements in $\langle f, g\rangle$ by [3, Theorem 5.1.2]. Since $fg \neq gf$, g is not parabolic by [3, Theorem 4.3.6]. Thus g is elliptic. By the structure of elementary groups [3, § 5.1], the set of multipliers $\mathrm{S} = \{1, \omega, \omega^2, \dots, \omega^{q-1}\}$ where $\omega = \exp(2\pi i/q)$, $0 \le q \le 6$, $q \neq 5$. So g is elliptic of order less than or equal to six. Therefore,

$$\max\{d(f), d(g)\} \ge d(g) \ge 2\sin(\pi/6) = 1. \ \square$$

4.18. Corollary. *Suppose that $\langle f, g\rangle$ is a discrete subgroup of* $\mathbf{M}$ *and that f is parabolic. If $\gamma(f,g) \neq 0$, then*

$$\max\{d(f), d(g)\} \ge c_2, \qquad 1.347 \le c_2 \le 1.6.$$

Proof. It follows from the proof of Lemma 4.17 that $\max\{d(f), d(g)\} \ge 1.347$. The subgroup generated by

$$f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

is discrete. We have $d(f) = d(g) = 8/5$. $\square$

4.19. Example. Let $f = z + 1/m$, $g = z + 1/n$. Then $\langle f, g\rangle$ is a discrete group with $fg = gf$. We have $d(f) \to 0$, $d(g) \to 0$, as $m, n \to \infty$.

4.20. Lemma. *Suppose that $\langle f, g\rangle$ is a discrete subgroup of* $\mathbf{M}$ *and that f is loxodromic. If $fg \neq gf$, then*

$$\max\{d(f), d(g)\} \ge c_3, \qquad .863 \le c_3 \le .911\dots .$$

Proof. By Lemma 4.15 and Lemma 4.17, we may assume that g is also loxodromic. Thus $\gamma(f,g) \neq \beta(f)$ by [10, Lemma 2.31]. We also have $\gamma(f,g) \neq 0$

since otherwise f and g have one common fixed point, and hence two common fixed points by [3, Theorem 5.1.2]. Thus $fg = gf$, a contradiction.

Since f is loxodramic, it is conjugate to a mapping of the form $w = \rho^2 e^{i2\theta} z$, $-\pi/2 < \theta \leq \pi/2$, $0 < \rho \neq 1$.

By [9, Theorem 2.7 and Theorem 3.11],

$$2|\beta(f)| \leq m(f)^2 \leq 8\cos^2\theta \frac{d(f)^2}{4 - d(f)^2}. \tag{4.21}$$

Let $c_0 = 2\bigl(\cos(2\pi/7) + \cos(\pi/7) - 1\bigr)$, $d_0 = 2 - 2\cos(\pi/7)$. It follows from (4.21) that if $|\beta(f)| \geq c_0$, then

$$d(f) \geq 2\Bigl(\frac{\cos(2\pi/7) + \cos(\pi/7) - 1}{\cos(2\pi/7) + \cos(\pi/7) + 1}\Bigr)^{1/2} = .911\ldots .$$

Suppose that $\max\{|\beta(f)|, |\beta(g)|\} \leq c_0$. Let $\gamma = \gamma(f,g)$, $\beta = \beta(f)$, $v = \cos^2\theta$, $a = 0.21$. If $|\gamma| \geq a$, then by [9, Theorem 2.7 and Corollary 3.15],

$$\max\{d(f), d(g)\} \geq \Bigl(\frac{4|\gamma|^{1/2}}{|\gamma|^{1/2} + 2}\Bigr)^{1/2} \geq .863.$$

We now assume that $|\gamma| \leq a$. Then $|\beta| \geq 1 - a$ by Jørgensen's inequality (4.1). Let $g_1 = gf^{-1}g^{-1}fgfg^{-1}f^{-1}g$. If $\gamma(f, g_1) \neq 0$ and $\gamma(f, g_1) \neq \beta(f)$, then

$$|\gamma(f, g_1)| = \bigl|\gamma\bigl(\gamma^2 - (\beta - 1)\gamma - (\beta - 1)\bigr)^2\bigr| \geq d_0$$

by [5, Corollary 3.8]. It follows that

$$|\beta - 1| > \frac{1}{1 + a}(\sqrt{d_0/a} - a^2). \tag{4.22}$$

Since $1 - a \leq |\beta| \leq c_0$ and

$$|\beta - 1|^2 = -16v^2 + 4(4 - |\beta|)v + (|\beta| + 1)^2,$$

it follows from (4.22) that $v < .971$. If $m(f) \geq m(g)$, then

$$1.78 \leq m(f)m(g) \leq m(f)^2 \leq \frac{8vd(f)^2}{4 - d(f)^2}$$

by Lemma 4.3 and (4.21). Therefore,

$$\max\{d(f), d(g)\} \geq d(f) \geq .863.$$

If $\gamma^2-(\beta-1)\gamma-(\beta-1)=0$, then $\beta=(1+\gamma+\gamma^2)/(1+\gamma)$. Since $|\gamma|\le a$,

$$\frac{1}{|\beta|}\le \max_{|z|=a}\left|\frac{1+z}{1+z+z^2}\right|<1.05.$$

It follows from (4.21) that $d(f)>.877$.

We finally show that $\gamma(f,g_1)=\beta(f)$ can not occur. By [10, Lemma 2.29], there exists an elliptic h of order two such that $\langle f,h\rangle$ is discrete with $\gamma(f,h)=\gamma(f,g)$. Let $g_2=hf^{-1}h^{-1}fhfh^{-1}f^{-1}h$. Suppose that $\gamma(f,g_1)=\beta(f)$. We will obtain a contradiction. Since $\gamma(f,g_2)=\gamma(f,g_1)=\beta(f)$, g_2 is of order two by [10, Lemma 2.31]. After a conjugation, we may assume that f and h are represented by

$$A=\begin{pmatrix} u & 0\\ 0 & 1/u\end{pmatrix},\qquad B=\begin{pmatrix} e_{11} & e_{12}\\ e_{21} & e_{22}\end{pmatrix}.$$

Thus $\beta=(u-1/u)^2$, $\gamma=-e_{12}e_{21}(u-1/u)^2$. Elementary calculations show that

$$BA^{-1}B^{-1}ABAB^{-1}A^{-1}B=\\ \begin{pmatrix} e_{11}((\gamma+1)^2-\gamma u^{-2}) & e_{12}(\gamma^2-(\beta-1)\gamma-(\beta-1))\\ e_{21}(\gamma^2-(\beta-1)\gamma-(\beta-1)) & e_{22}((\gamma+1)^2-\gamma u^2)\end{pmatrix}.$$

Since $hf^{-1}h^{-1}fhfh^{-1}f^{-1}h$ is of order two,

$$e_{11}\big((\gamma+1)^2-\gamma u^{-2}\big)+e_{22}\big((\gamma+1)^2-\gamma u^2\big)=0. \tag{4.23}$$

Notice that $e_{11}+e_{22}=0$ (h is of order two). If $e_{11}\ne 0$, then (4.23) implies that $\beta(\beta+4)=0$, a contradiction. If $e_{11}=0$, then $e_{12}e_{21}=-1$. Hence $\gamma=\beta$, another contradiction.

The number $.911\ldots$ occurs in the $(2,3,7)$ triangle group (see [9]). □

4.24. Example. Let $f=(1+1/m)z$, $g=(1+1/n)z$. Then $\langle f,g\rangle$ is discrete and $fg=gf$. We have $d(f)\to 0$, $d(g)\to 0$, as $m,n\to\infty$.

4.25. Theorem. *Suppose that $\langle f,g\rangle$ is a discrete subgroup of $\mathbf{M}$. If $fg\ne gf$, then*

$$\max\{d(f),d(g)\}\ge c,\qquad .863\le c\le .911\ldots.$$

Proof. This follows from Lemma 4.15, Lemma 4.17 and Lemma 4.20. □

4.26. Corollary. *Suppose that $\langle f,g\rangle$ is a discrete subgroup of $\mathbf{M}$. If $\gamma(f,g)\ne 0$ and g is not of order two, then*

$$\max\{d(f,g),d(f,g^{-1})\}\ge .863.$$

Proof. This follows from the fact $d(f,g)=d(fg^{-1})$ and Theorem 4.25. □

4.27. Theorem. *Suppose that G does not have a G-orbit in $\overline{\mathbf{C}}$ that has less than three points. Then G is discrete if and only if for each pair $f, g \in G \setminus \{\mathrm{id}\}$, either $fg \neq gf$ or*

$$\max\{d(f), d(g)\} \geq .863.$$

Proof. By Theorem 4.25, it suffices to prove that G is discrete if for each pair f, $g \in G \setminus \{\mathrm{id}\}$, either $\max\{d(f), d(g)\} \geq .863$ or $fg = gf$. Suppose that G is not discrete. We will obtain a contradiction. Since G is not discrete, there exists distinct elements f_1, f_2, $\cdots (\neq \mathrm{id})$ in G such that $d(f_n, \mathrm{id}) \to 0$ as $n \to \infty$. So there exists an $N > 0$ such that $d(f_n, \mathrm{id}) < 1/2$ if $n \geq N$. Thus for all m, $k \geq N$, $f_m f_k = f_k f_m$ by hypothesis. So we may assume that there exists a sequence $\{f_n\}$ such that

$$\begin{aligned} d(f_n, \mathrm{id}) &\to 0, && \text{as } n \to \infty \\ f_m f_k &= f_k f_m, && \text{for all } m, k \geq 1. \end{aligned}$$

Furthermore, since all f_i's are distinct and $d(f_n, \mathrm{id}) \to 0$, by passing to a subsequence of $\{f_n\}$ if necessary, we may assume that f_k is not of order two for all $k \geq 1$. Since f_k and f_m commute for all k, $m \geq 1$, $\mathrm{fix}(f_k) = \mathrm{fix}(f_m)$ by [3, Theorem 4.3.6]. Let $\mathrm{fix}(f_k) = \{a, b\}$ for all $k \geq 1$ (it is possible that $a = b$). For any $g \in G \setminus \{\mathrm{id}\}$, $d(g f_n g^{-1}) \to 0$, as $n \to \infty$. So there exists an n_0 such that $\max\{d(g f_n g^{-1}), d(f_n)\} \leq 1/2$ if $n \geq n_0$. By hypothesis, $g f_{n_0} g^{-1} f_{n_0} = f_{n_0} g f_{n_0} g^{-1}$. Since f_{n_0} is not of order two, $\mathrm{fix}(g f_{n_0} g^{-1}) = \mathrm{fix}(f_{n_0}) = \{a, b\}$ by [3, Theorem 4.3.6]. Thus $g\{a, b\} = \{a, b\}$. Since g is arbitrary, $\cap_{h \in G} h(a) \subset \{a, b\}$. This contradicts the assumption that G does not have a G-orbit that has less than three points. □

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