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SOME REMARKS ON NON-DISCRETE MÖBIUS GROUPS

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Abstract. This paper contains some tentative steps towards describing the structure of non-discrete subgroups of $\mathrm{SL}(2,\mathbf{R})$. The main idea is that if a one-parameter family of groups G_z varies analytically with the parameter z, then, using analytic continuation, certain results about discrete groups can be analytically continued to those groups in the family that are not discrete. The paper concentrates on families of groups generated by two parabolic transformations and, as an illustration, contains a proof that, for all but a countable set of exceptional values of the parameter z, the hyperbolic area of a hyperbolic quadilateral whose sides are paired by some pair of parabolic generators of G_z is independent of the choice of generators. This is the analogue of the familiar result that the area of a fundamental region of a discrete group is independent of the choice of the generators, but it applies here to almost all non-discrete groups in the family. It is also shown that exceptional groups exist, and explicit examples of these are given. The paper ends with some unanswered questions.

1. Introduction

While the general structure of discrete subgroups of $\mathrm{SL}(2,\mathbf{C})$ is well understood, very little has been written about its non-discrete subgroups. This paper contains some tentative steps in this direction. Let G_z be a one-parameter family of subgroups of $\mathrm{SL}(2,\mathbf{C})$ (with the parameter z lying in some domain D in the complex plane), suppose that the groups G_z vary analytically with z, and also that, for a range of values of z, the G_z are discrete groups. We shall show how we can then use analytic continuation to transfer certain facts about the discrete groups to almost all non-discrete groups in the family. This transfer is encapsulated in the uniqueness principle given in Section 2, and the ideas here have been motivated by the work in [7] and [9].

Most of this paper is about parabolic Möbius transformations. In Section 3 we apply these ideas to a family of groups generated by two parabolic transformations. Numerous papers have been written on this family, and here we investigate when two given parabolic elements in G generate G, and we include a discussion of some of the exceptional groups in the family. In Section 4, we make some observations regarding the real groups within this family.

As usual, we shall change freely between matrices in $\mathrm{SL}(2,\mathbf{C})$ and their action as Möbius maps, namely elements $\mathrm{PSL}(2,\mathbf{C})$ on the complex sphere, and we use I

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to denote both the unit matrix and the identity map. We shall assume a knowledge of the basic theory of discrete Möbius groups for which the reader is referred to, for example, [1], [3], [4], [6] and [8]. In particular, we recall that $\mathrm{PSL}(2,\mathbf{C})$ acts on the upper-half space $\mathscr{H}^3$ of $\mathbf{R}^3$ as the group of conformal isometries of hyperbolic 3-space.

To emphasize the difference between discrete and non-discrete subgroups of $\mathrm{SL}(2,\mathbf{C})$, we recall that a discrete subgroup G is a closed subset of $\mathrm{SL}(2,\mathbf{C})$ containing only isolated points, whereas the generic non-discrete subgroup of $\mathrm{SL}(2,\mathbf{C})$ is dense in $\mathrm{SL}(2,\mathbf{C})$. For a precise statement and proof of this, see [2].

2. The uniqueness principle

We begin with a general principle based on analytic continuation. Let D be a domain in the complex plane, let $\mathscr{R}$ be the ring of functions analytic in D, and let $\mathrm{SL}(2,\mathscr{R})$ be the group of 2×2 matrices with entries in $\mathscr{R}$ and with determinant 1 (throughout D). For each z in D, and each X in $\mathrm{SL}(2,\mathscr{R})$, we can evaluate the entries of X at z and so obtain an element X_z of $\mathrm{SL}(2,\mathbf{C})$. For each z, the *evaluation map* $e_z\colon X\mapsto X_z$ is a homomorphism of $\mathrm{SL}(2,\mathscr{R})$ into $\mathrm{SL}(2,\mathbf{C})$.

Now consider a finitely generated (hence countable) subgroup $\mathscr{G}$ of $\mathrm{SL}(2,\mathscr{R})$, restrict the map e_z to $\mathscr{G}$, and let G_z be the image group $e_z(\mathscr{G})$. The kernel of e_z is $\{X\in\mathscr{G}: X_z=I\}$ and, as $\mathscr{G}$ is countable and the entries in X are analytic functions, this is non-trivial for only a countable set of z in D. Thus, apart from a countable set of z, $e_z\colon\mathscr{G}\to G_z$ is an isomorphism. We shall say that z in D is *exceptional* if e_z is not an isomorphism, and we denote the countable set of exceptional z by $\mathscr{E}$. Of course, $\mathscr{E}$ depends on $\mathscr{G}$ but we omit this dependence from our notation. The main result in [9] says that $\mathscr{E}$ is dense in some neighbourhood of any z for which G_z is not discrete, and also in some neighbourhood of any z in $\mathscr{E}$.

Our basic result is the following simple

Uniqueness principle. *Given $\mathscr{G}$, suppose that there is an arc σ in D such that for each z in σ, G_z is discrete, non-elementary and geometrically finite. If X and Y are in $\mathscr{G}$, and, if for some w in σ, $X_w=Y_w$ then $X=Y$ in $\mathrm{SL}(2,\mathscr{R})$; in particular, $X_z=Y_z$ for all z in D.*

The uniqueness principle can be motivated by the following geometric argument. First, it suffices to consider the case when $Y=I$, for we can then apply the result with X replaced by XY^{-1}. For each z in σ the group G_z acts discontinuously in $\mathscr{H}^3$ and we can find a fundamental domain F_z for its action in $\mathscr{H}^3$. Suppose now that F_z can be chosen to vary continuously with z; more precisely, that F_z can be chosen so that for z in some σ-neighbourhood N of w (that is, N is a relatively open subset of σ that contains w),

$$\mathrm{interior}\Big(\bigcap_{z\in N} F_z\Big)\neq\emptyset. \tag{2.1}$$

Denote this open set by V, and choose a point ζ in V. The hypothesis $X_w = I$ implies that $X_w(\zeta) = \zeta \in V$, and the continuity of $z \mapsto X_z(\zeta)$ ensures that $X_z(\zeta) \in V$ for all z in some σ-neighbourhood N_1 of w. This implies that for all z in N_1, $X_z(F_z) \cap F_z \neq \emptyset$ and hence (because F_z is a fundamental domain for G_z) that $X_z = I$. As $X_z = I$ for all z in N_1, analytic continuation ensures that $X_z = I$ for all z in D.

In many cases, we can find such fundamental domains F_z explicitly, and in these cases the uniqueness principle is automatically applicable (because this geometric argument is sufficient for the proof). It is, however, also possible to give an analytic proof of the uniqueness principle (leaving the existence of σ unresolved) based on Jorgensen's inequality [5]: if A and B are in $\mathrm{SL}(2, \mathbf{C})$, and if $\langle A, B\rangle$ is discrete and non-elementary, then

$$(2.2) \qquad |\mathrm{trace}\,(ABA^{-1}B^{-1}) - 2| + |\mathrm{trace}^2(A) - 4| \geq 1.$$

The proof of the uniqueness principle. We suppose that $X_w = I$, but that $X \neq I$ (in $\mathscr{G}$); then w is an isolated point of $\{z \in D : X_z = I\}$. Now select Y and W in $\mathscr{G}$ such that Y_w and W_w are loxodromic with no common fixed points. Then there are open neighbourhoods N_1, N_2 and N_3 of w, all lying in D, such that

(1) $z \in N_1$ and $z \neq w$ implies $X_z \neq I$;
(2) $z \in N_2$ implies Y_z and W_z are loxodromic with no common fixed point;
(3) $z \in N_3$ implies (by continuity)

$$|\mathrm{trace}^2(X_z) - 4| < \tfrac{1}{2}, \quad |\mathrm{trace}\,(X_z Y_z X_z^{-1} Y_z^{-1}) - 2| < \tfrac{1}{2},$$

$$|\mathrm{trace}\,(X_z W_z X_z^{-1} W_z^{-1}) - 2| < \tfrac{1}{2}.$$

Now take z in $\sigma \cap N_1 \cap N_2 \cap N_3$ with $z \neq w$. Then $\langle X_z, Y_z\rangle$ is discrete and, from (3) and (2.2), it is elementary. With (1), this means that X_z and Y_z have the same fixed point set (or X_z is elliptic of order 2, but this implies that $\mathrm{trace}\,(X_z) = 0$ and this happens only countably often). Exactly the same argument holds for W_z instead of Y_z; thus Y_z and W_z have the same fixed point set, contrary to (2). This shows that w is a non-isolated point of $\{z \in D : X_z = I\}$ and the proof is complete.

We end this section with a few simple illustrations of the use of the uniqueness principle. First, as a trivial application, we take z in σ, X in the kernel of e_z, and $Y = I$. As $X_z = I = Y_z$ we have $X = I$ throughout D; thus for all z in σ, e_z is an isomorphism; we deduce that $\mathscr{E} \cap \sigma = \emptyset$.

An element X of $\mathrm{SL}(2, \mathscr{R})$ is said to be *parabolic* if its trace is constant in D with value ± 2; then, for each z, X_z is a parabolic element of $\mathrm{SL}(2, \mathbf{C})$. Of course, there may be *spurious parabolic elements* in $\mathscr{G}$, namely elements Y for which Y_z

is parabolic for some, but not all, z. Suppose now that X_z is parabolic. Then either X has constant trace (so that X is parabolic) or z lies in the countable set of solutions of $\operatorname{trace}^2(X_z)=4$. As $\mathscr{G}$ is countable, we thus see that there is a countable subset $\mathscr{P}$ of D such that these spurious parabolic elements exist in G_z only when $z\in\mathscr{P}$; equivalently, if $z\notin\mathscr{P}$ then X_z is parabolic if and only if X is. The relationship between $\mathscr{E}$ and $\mathscr{P}$ is not clear and seems worth exploring.

Now let us now consider the stabiliser of a parabolic fixed point. First, take z outside of the countable set $\mathscr{E}\cup\mathscr{P}$, and take w in σ but outside of $\mathscr{E}\cup\mathscr{P}$. Now suppose that ζ is fixed by some parabolic element of G_z, and let Γ_z be the subgroup of all parabolic elements of G_z that fix ζ. Then each element of Γ_z is the evaluation of some parabolic element in $\mathscr{G}$ and so, under the isomorphism $e_w^{-1}e_z$, Γ_z corresponds to an abelian subgroup, say Γ_w, of parabolic elements of G_w. As Γ_w is a free abelian group on one or two generators, so too is Γ_z, and this means that Γ_z is discrete. Thus for all but a countable set of points in D, *the stabiliser of a parabolic fixed point of G_z is discrete* even though G_z itself may not be. Clearly, a similiar statement can be made about the subgroup of loxodromic elements in G_z with a common pair of fixed points.

Finally, it is a direct consequence of the Uniqueness Principle that if a discrete group G_w, where w is in σ but not in $\mathscr{E}\cup\mathscr{P}$, has only a finite number of conjugacy classes of purely parabolic subgroups (a familiar situation in the theory of discrete groups) then this property is transmitted to $\mathscr{G}$, and then on to G_z for all z outside the countable set $\mathscr{E}\cup\mathscr{P}$ regardless of whether G_z is discrete or not.

3. An example

We shall now apply these ideas to the example in which D is the complex plane, $\mathscr{R}$ is the ring $Z[\tau]$ of complex polynomials with integer coefficients and indeterminate τ, σ is the real interval $(2,+\infty)$, and $\mathscr{G}$ is the subgroup of $\mathrm{SL}(2,\mathscr{R})$ generated by the two parabolic matrices

$$A=\begin{pmatrix}1&\tau\\0&1\end{pmatrix},\qquad B=\begin{pmatrix}1&0\\\tau&1\end{pmatrix}. \tag{3.1}$$

The Uniqueness Principle is applicable to this family; indeed, by considering the isometric circles of B_t and B_t^{-1} (of radius $1/|t|$), it is clear that, by choosing F_t to be the obvious Ford fundamental region, namely

$$F_t=\{z:|\operatorname{Re}(z)|<\tfrac{1}{2}t,\ |tz+1|>1,\ |tz-1|>1\},$$

the condition (2.1) holds when $t>2$. For these t (as is evident from other considerations), $e_t\colon\mathscr{G}\to G_t$ is an isomorphism.

We consider the problem of when two parabolic elements U_t and V_t generate G_t. Of course, this is so if U_t and V_t are jointly conjugate to A_t and B_t

(that is, if there is some X_t in G_t with $U_t = X_t A_t X_t^{-1}$ and $V_t = X_t B_t X_t^{-1}$), but the question is whether or not there are pairs of parabolic generators *not* of this form. We shall show that *there can be other pairs*, but only for a countable set of exceptional t. Our proof is based on the following result.

Theorem 3.1. *Suppose that U and V are non-commuting parabolic elements of $\mathscr{G}$. Then the following are equivalent:*
(a) *U and V generate $\mathscr{G}$;*
(b) *$\operatorname{trace}[U,V] = 2+\tau^4$, where $[U,V]$ is the commutator $UVU^{-1}V^{-1}$;*
(c) *U and V are jointly conjugate in $\mathscr{G}$ to A and B.*

For all but a countable set of t, e_t is an isomorphism of $\mathscr{G}$ onto G_t which preserves parabolicity (in both directions) so that Theorem 3.1 enables us to derive the following information about the *generic non-discrete subgroup* G_t of $\mathrm{SL}(2,\mathbf{C})$.

Theorem 3.2. *For all but a countable set of t, the non-commuting parabolic elements U_t and V_t generate G_t if and only if $\operatorname{trace}[U_t,V_t] = 2+t^4$.*

Before proving Theorem 3.1, we show that the conclusion of Theorem 3.2 can fail for some exceptional values of t. For any positive integer p,

$$A^pB^{-p(p+1)}A = \begin{pmatrix} 1-p^2(p+1)\tau^2 & (p+1)\tau - p^2(p+1)\tau^3 \\ -p(p+1)\tau & 1-p(p+1)\tau^2 \end{pmatrix}.$$

We now put $\tau = 1/p$ and (for simplicity) omit the suffix $1/p$ on A and B; thus from now on, but in this example only, we have

$$A = \begin{pmatrix} 1 & 1/p \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 1/p & 1 \end{pmatrix}.$$

Then

$$A^pB^{-p(p+1)}A = -\begin{pmatrix} p & 0 \\ p+1 & p^{-1} \end{pmatrix} = C,$$

say, is a hyperbolic element fixing 0. It is easy to check that, for every real s,

$$CB^sC^{-1} = C\begin{pmatrix} 1 & 0 \\ s/p & 1 \end{pmatrix}C^{-1} = \begin{pmatrix} 1 & 0 \\ s/p^3 & 1 \end{pmatrix} = B^{s/p^3},$$

and, as a consequence of this, that

$$C^nB^sC^{-n} = B^{s/p^{3n}}.$$

Taking $s=1$ we find that

$$\langle A,B\rangle \subset \langle A, B^{1/p^{3n}}\rangle \subset \langle A,B,C\rangle \subset \langle A,B\rangle,$$

so that A and $B^{1/p^{3n}}$ are a pair of parabolic generators of G. However, as

$$\operatorname{trace}[A, B^{1/p^{3n}}] = 2 + \frac{1}{p^{6n+4}} \to 2, \tag{3.1}$$

as $n\to\infty$, these generators are not jointly conjugate to A and B.

In preparation for our proof of Theorem 3.1, recall that U and V are jointly conjugate to P and Q in $\mathcal{G}$ if there is some W in $\mathcal{G}$ with $U = WPW^{-1}$ and $V = WQW^{-1}$. It is often desirable to replace an element by its inverse and, to avoid frequent reference to this, we henceforth say that U and V are *jointly conjugate* to P and Q if there is some W with $U = WP^{\varepsilon}W^{-1}$ and $V = WQ^{\delta}W^{-1}$, where $\varepsilon = \pm 1$ and $\delta = \pm 1$. In most cases, this is all that matters.

The proof of Theorem 3.1. First, we verify (c) *implies* (b). A calculation shows that $\operatorname{trace}[A_t, B_t] = 2 + t^4$, so that if (c) holds, then

$$\operatorname{trace}[U_t, V_t] = \operatorname{trace}[A_t, B_t] = 2 + t^4.$$

As this holds for all t, (b) follows.

Conversely, let U and V be parabolic elements of $\mathcal{G}$ satisfying (b). Then, for any positive t, U_t and V_t have distinct fixed points (else they commute and $\operatorname{trace}[U_t, V_t] = 2$) and so, by moving the fixed points to 0 and ∞, we see that they are jointly conjugate to a pair of matrices

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix},$$

respectively, where $\alpha > 0$. We deduce that

$$2 + \alpha^4 = \operatorname{trace}[U_t, V_t] = 2 + t^4;$$

thus $\alpha = t$ and for some W_t, $U_t = W_t A_t W_t^{-1}$ and $V_t = W_t B_t W_t^{-1}$. As this holds for $t = 3$, say, the Uniqueness Principle ensures that this joint conjugacy is transferred from G_3 to $\mathcal{G}$ and (c) holds. This proves that (b) and (c) are equivalent.

Next, if (c) holds, say $U = WAW^{-1}$ and $V = WBW^{-1}$, where $W \in \mathcal{G}$, then U and V generate $W\mathcal{G}W^{-1}$, which is $\mathcal{G}$, so that (a) holds.

Finally, we show that (a) implies (c). Suppose, then, that U and V are a pair of parabolic generators of $\mathcal{G}$ and apply the isomorphism e_3 (that is, put $\tau = 3$). Then U_3 and V_3 are parabolic generators of G_3 and so, from the standard theory of Fuchsian groups, each is conjugate in G_3 to some power of A_3 or B_3. As $X_3 = Y_3$ in G_3 if and only if $X = Y$ in G, we can take the liberty of omitting the suffix 3 throughout the following argument; thus in what follows (and until further notice), A is really A_3 and so on.

By considering WGW^{-1} ($= G$), where $W \in G$, we may suppose that

$$U = A^p, \qquad V = XC^qX^{-1},$$

where $X \in G$, and C is A or B, and p and q are integers. By a further conjugation with respect to some A^k, we may assume that the fixed point ζ of V

lies in the interval $|x| < 3/2$ (recall that $A(z) = z + 3$). Let Σ be the region lying outside the isometric circles of B and B^{-1} and in the strip $-3/2 < x < 3/2$; then Σ is a fundamental region for G (generated by A and B).

Next, all elements of G that do not fix ∞ have an isometric circle whose radius is no larger than that for B; thus, as the fixed point of the parabolic V lies in the interval $|x| < 3/2$, we find

(i) that the region Σ' lying outside the isometric circles of V and V^{-1} and in the strip $-p/2 < x < p/2$ is a fundamental region for G (generated by U and V), and

(ii) that $\Sigma \subset \Sigma'$.

It follows that $\Sigma = \Sigma'$, and hence that $p = \pm 1$. Replacing U by U^{-1} if necessary, we may now assume that $U = A$. Next, as $\Sigma = \Sigma'$, we see that V fixes the origin and so is a power of B, say $V = B^q$. Finally, as the isometric circles of V and B have the same radius, we see that $q = \pm 1$ and, replacing V by V^{-1} if necessary, we have $V = B$.

We now revert to the inclusion of the suffix t. The argument above shows that given a pair U and V of parabolic generators of $\mathscr{G}$, there is some Y in $\mathscr{G}$ such that

$$U_3 = Y_3 A_3 Y_3^{-1}, \qquad V_3 = Y_3 B_3 Y_3^{-1}.$$

The Uniqueness Principle now shows that $U = YAY^{-1}$ and $V = YBY^{-1}$ in $\mathscr{G}$, and this is (a). The proof is complete.

4. Subgroups of $\mathrm{SL}(2, \mathbf{R})$

We continue with the discussion of the group $\mathscr{G}$ generated by A and B in (3.1), and we now focus our attention on real values of t. First, it is easy to see that $\mathscr{E}$ is dense in $(0, 2)$; indeed, for a dense set of t in $(0, 2)$, the matrix $A_t B_t^{-1}$ is a rotation of the hyperbolic plane of finite order, so that these values of t are in $\mathscr{E}$. Moreover, $A_t B_t^{-1}$ is parabolic when, and only when, $t^2 = 4$ so this is an example of a spurious parabolic element of $\mathscr{G}$. Finally, note that $\mathscr{P} \cap (0, 2) = \emptyset$.

Our earlier results suggest that it might be profitable to consider

$$\mathscr{T}(G) = \{\mathrm{trace}\,[U, V] : U, V \text{ non-commuting parabolic elements of } G\},$$

where G is a finitely generated non-elementary subgroup of $\mathrm{SL}(2, \mathbf{C})$, and we shall now show that, for a subgroup G of $\mathrm{SL}(2, \mathbf{R})$, $\mathscr{T}(G)$ is sufficient to determine whether G is discrete or not.

Theorem 4.1. *Let G be a non-elementary subgroup of* $\mathrm{SL}(2, \mathbf{R})$. *Then*

(1) *G is discrete if and only if $\mathscr{T}(G)$ is a discrete subset of $[3, +\infty)$;*

(2) *G is non-discrete if and only if $\mathscr{T}(G)$ is dense in $[2, +\infty)$.*

Apart from countable many exceptional values, for all t in $(0,2)$, G_t is non-discrete and the set $\mathscr{T}(G_t)$ is dense in $[2,+\infty)$. In view of this, Theorem 3.2 is somewhat surprising in that it says that in these cases we can actually characterise the pairs of parabolic generators of G_t by the trace of their commutator.

Proof of Theorem 4.1. Suppose first that G is discrete, and that U and V are non-commuting parabolic elements of G. By conjugation, we may suppose that U fixes ∞, V fixes the origin, and that the stabiliser of ∞ is generated by $z \mapsto z+1$. It follows that U and V are of the form

$$U = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}, \tag{4.1}$$

where a is a non-zero integer. As G contains $z \mapsto z+1$, Jorgensen's inequality yields $|b| \geq 1$, so that

$$\operatorname{trace}[U,V] = 2 + a^2b^2 \geq 3.$$

As the set of traces of all elements of a discrete group G is a discrete set ([4]), this establishes one implication in (1). Notice that this argument shows that $\operatorname{trace}[U,V] > 2$ for *any* pair of non-commuting parabolic elements of $\mathrm{SL}(2,\mathbf{C})$.

To complete the proof of (1) we shall now show that if $\inf \mathscr{T}(G) > 2$, then G is discrete. We suppose, then, that there is some positive ε such that

$$\inf \mathscr{T}(G) = 2 + \varepsilon^4 > 2$$

(so that G does have parabolic elements). Now consider a parabolic element U in G. As G is non-elementary, it contains an element X which does not share a fixed point with A, so that V defined by $V = XAX^{-1}$ is a parabolic element of G which does not commute with U. By conjugation, we may suppose that

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad U^s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix},$$

where $ad - bc = 1$ and $c \neq 0$, and s is any real number (though U^s is not necessarily in G). A computation yields

$$2 + c^4 = \operatorname{trace}[U,V] \geq 2 + \varepsilon^4;$$

more generally, if s is such that $U^s \in G$, then

$$2 + s^2c^4 = \operatorname{trace}[U^s,V] \geq 2 + \varepsilon^4$$

so that

$$|s|c^2 \geq \varepsilon^2. \tag{4.2}$$

Fixing X and varying s, (4.2) shows that there is a smallest (Euclidean) translation in G, and hence that the subgroup of translations in G is cyclic and discrete. Further, it is well known that this would be false if G contained a hyperbolic element fixing ∞; thus the stabiliser $G(\infty)$ of ∞ is a discrete group of translations. Now let the smallest translation in G be $z \mapsto z + s_0$, where $s_0 > 0$. This, together with (4.2), now shows that $|c| \geq \varepsilon/s_0$ for every choice of X, and with this it is clear that there is some horocycle $\mathscr{H}$ at ∞ which is mapped onto itself by every element of $G(\infty)$ and to a disjoint horocycle by all other elements of G. This proves that G is discrete and completes the proof of (1).

To complete the proof of (2), it remains only to assume that G is non-elementary and non-discrete, and to show that $\mathscr{T}(G)$ is dense in $[2, +\infty)$. Let

$$\Lambda = \{x : 2 + x \in \mathscr{T}(G)\}.$$

We have seen above that the set $\mathscr{T}(G)$ accumulates at 2, so

(a) Λ contains a sequence x_n decreasing strictly to zero.

Also, if A and B are parabolic and $\operatorname{trace}[A, B] = 2 + x$, then $\operatorname{trace}[A^n, B] = 2 + n^2 x$ as can be seen by assuming (after conjugation) that $A(z) = z + 1$. Thus

(b) if $x \in \Lambda$ then $n^2 x \in \Lambda$ for $n = 1, 2, \dots$.

It is easy to see that (a) and (b) imply that Λ is dense in $(0, +\infty)$ as follows. Take any interval $I = (a, b)$, where $0 < a < b$. Then, for all sufficiently large n, the union of intervals

$$\Big(\frac{a}{n^2}, \frac{b}{n^2}\Big) \cup \Big(\frac{a}{(n+1)^2}, \frac{b}{(n+1)^2}\Big) \cup \cdots \tag{4.3}$$

is connected and so contains some interval of the form $(0, \eta)$, where $\eta > 0$. From (a), there is some x in $\Lambda \cap (0, \eta)$, so that x lies in one of the intervals in (4.3). It follows that for some k, $k^2 x \in I$ and so $I \cap \Lambda \neq \emptyset$ as required. The proof of Theorem 4.1 is complete.

5. Closing remarks

For elements in $\mathrm{SL}(2, \mathbf{R})$, the geometric interpretation of the trace of the commutator is of interest. Still discussing the group generated by A and B in (3.1), it is evident that when $0 < t < 2$, the trace of $[A_t, B_t]$ determines, and is determined by, the hyperbolic area of the polygon bounded by the lines $x = \pm t/2$ and the isometric circles of B_t and B_t^{-1} (a similar statement is true when $t > 2$, providing the area in question is taken to be the intersection of F_t with the Nielsen convex region for G_t). Of course, for the generic t in $(0, 2)$, G_t does not have a fundamental polygon (for it is not discrete) but we can still interpret Theorem 3.2 as saying that, for all but a countable set of t, the hyperbolic area of a hyperbolic quadilateral whose sides are paired by some pair of parabolic generators of G_t is *independent of the choice of generators.* This is the analogue of the familiar result that the area of a fundamental region of a discrete group is independent of the choice of the generators, but it applies here to almost all non-discrete groups in the family. For the exceptional groups discussed in Sections 3 and 4 (in which $\tau = 1/p$), we found pairs of parabolic generators P_n and Q_n with $\operatorname{trace}[p_n, Q_n] \to 0$. This shows that $G_{1/p}$ can be generated by two parabolic elements which are side-pairings of a hyperbolic quadilateral *whose area can be taken to be as small as we choose.*

Another geometric interpretation of the trace of the commutator of two parabolic elements is possible, and this does not require $0 < t < 2$. Let U and V be non-commuting parabolic isometries of the hyperbolic plane (with hyperbolic metric ϱ) and let ℓ be the hyperbolic geodesic joining the fixed points of U and V. The function

$$z \mapsto \sinh \tfrac{1}{2}\varrho(z, Uz)\ \sinh \tfrac{1}{2}\varrho(z, Vz)$$

attains its minimum $m(U,V)$ at each point of ℓ (and nowhere else) and moreover,

$$\operatorname{trace}[U,V] = 2 + 16m(U,V)^2. \tag{5.1}$$

Using this, it is possible to give alternative proofs of some of the earlier results. To prove (5.1), simply take $U(z) = z+1$ and $V(z) = (az+b)/(cz+d)$ and make the appropriate calculations (see [1, p. 200]).

Another curious feature of the example discussed above is that for certain values of t, B_t is conjugate to a power of itself. It is easy to see that this happens for a (general) parabolic element P only when P shares a common fixed point with some hyperbolic element; it is a well-known and frequently used fact that this never happens within a discrete group. There are many questions about the groups G_t studied above which may be of interest, and we mention a few here.

(1) A and B are not conjugate in $\mathscr{G}$, but can A_t and B_t ever be conjugate in G_t?
(2) Are there any t for which G_t contains infinitely many conjugacy classes of primitive parabolic elements?
(3) Is it possible for the stabiliser of, say, ∞ to be non-discrete yet not contain any hyperbolic elements?
(4) Is it possible for the stabiliser of, say, ∞ to contain a fractional power of A yet still be discrete?

Finally, we indicate briefly how the example discussed above might be generalised to, say, Schottky groups. We simply let A and B be two loxodromic elements of $\mathrm{SL}(2,\mathbf{C})$ with no common fixed points, and then embed A and B in a one parameter family of elements by, roughly speaking, shrinking the radii of their isometric circles to zero. For a range of values of the parameter w, the corresponding group G_w will be a Schottky group which, for these w, is the discrete free product of its two generators.

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