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ON THE EXISTENCE OF JENKINS–STREBEL DIFFERENTIALS USING HARMONIC MAPS FROM SURFACES TO GRAPHS

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Abstract. We give a new proof of the existence ([HM], [Ren]) of a Jenkins–Strebel differential Φ on a Riemann surface $\mathscr{R}$ with prescribed heights of cylinders by considering the harmonic map from $\mathscr{R}$ to the leaf space of the vertical foliation of Φ, thought of as a Riemannian graph. The novelty of the argument is that it is essentially Riemannian as well as elementary; moreover, the harmonic maps existence theory on which it relies is classical, due mostly to Morrey ([Mo]).

1. Introduction

In a series of pathbreaking papers ([Je], [Str1], [Str2]) in the 1950's and 1960's, Jenkins and Strebel proved the existence of holomorphic quadratic differentials on a Riemann surface $\mathscr{R}$, the complement of whose critical trajectories were Euclidean cylinders foliated by closed curves; the metrics associated to these differentials uniquely solved natural extremal length problems ([Je]), or the free homotopy classes of the core curves of the cylinders and the ratio of the moduli of the cylinders could be uniquely specified ([Str1]). Later, Renelt ([Ren]) and Hubbard and Masur ([HM]) showed that the homotopy classes of these core curves and the heights of the cylinders could also be specified. The goal of this note is to provide another proof of the existence of these differentials with prescribed heights on $\mathscr{R}$ based on energy-minimizing maps from $\mathscr{R}$ to graphs.

Our method is somewhat unusual in the subject of quadratic differentials on Riemann surfaces in that our techniques are essentially Riemannian, with the basic existence theory due mostly to Morrey [Mo] in 1948; the conformal type of the Riemann surface $\mathscr{R}$ is involved because of the conformal invariance of the total energy of the map. Our holomorphic quadratic differential on $\mathscr{R}$ is the Hopf differential, known classically (and emphasized recently by Schoen [S]) to result from a stationary point $I\colon \mathscr{R} \to N$ of a conformally invariant functional (here, it is important that I be stationary with respect to reparametrizations of the

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domain $\mathscr{R}$). A novel feature of our argument is its use of maps with domain $\mathscr{R}$, rather than, say, maps of cylinders into a range Riemann surface $\mathscr{R}$.

Here, roughly, is the proof. Draw the foliation with k prescribed cylinders on $\mathscr{R}$, and suppose for now that the leaf space is a boundaryless graph T (see Figure 1).

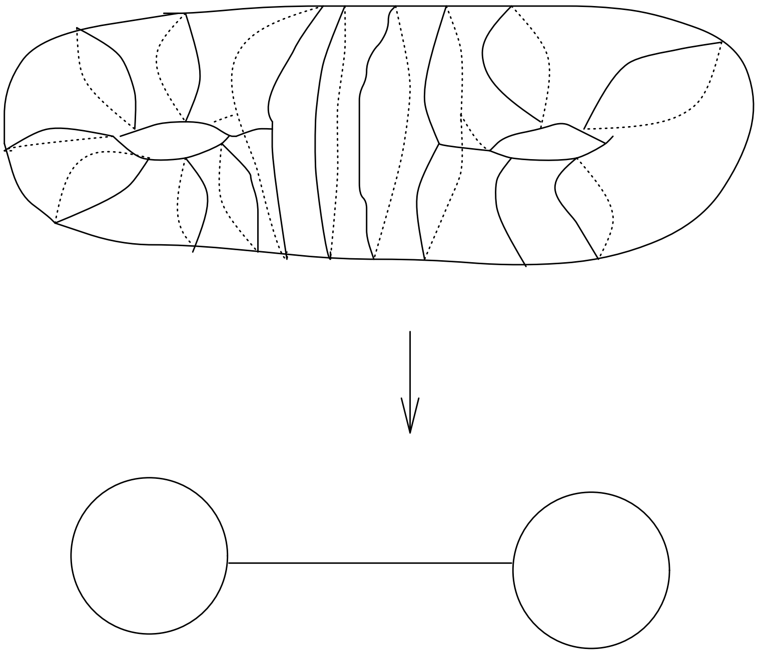

Figure 1.

The lengths of the graph are given by the heights of the corresponding cylinders. It then follows that there is a continuous energy-minimizing map $f\colon \mathscr{R} \to T$ in the homotopy class of the projection along leaves $\pi\colon \mathscr{R} \to T$: this follows because T is non-positively curved. (In this form, this result is a special case of the far deeper result of Schoen in [GS]. In the present situation, a more elementary proof is possible, and is effectively in the literature: we shall include a proof in the appendix for the sake of completeness.) The Hopf differential Φ is then holomorphic. Moreover, if the vector V is tangent to a regular vertical trajectory of Φ, then $Vf = 0$, from which it follows that f is constant along leaves of the vertical foliation of Φ. Since the homotopy class of f is non-constant, the leaves must be nowhere dense, and hence closed. It also follows that the map f is given precisely by projecting along the vertical leaves of the foliation of Φ to the (vertical) leaf space of Φ. Thus both f and π are defined by projecting along leaves of a (singular) foliation of $\mathscr{R}$ by closed leaves: we show that these two foliations are Whitehead equivalent (defined below) by using that f and π are homotopic to show that the vertical foliation of Φ has the same k cylinders of the same heights as $\mathscr{F}$. This concludes the argument in the case where the leaf space T of $\mathscr{F}$ is a boundaryless graph. In the case where T has boundary, we approximate $\mathscr{F}$ by a foliation $\mathscr{F}_\varepsilon$ whose graph T_ε is boundaryless (T_ε is constructed from T by attaching a small loop at each boundary point of T) and obtain the quadratic differential representative of $\mathscr{F}$ as a limit.

We organize this paper as follows. In the second section, we define our terms, set our notation and recall some background information. In Section 3, we prove the main result. The paper concludes with an appendix giving a reasonably elementary proof of the existence of harmonic maps from surfaces to boundaryless graphs.

2. Notation and background

2.1. Quadratic differentials. A measured foliation $(\mathscr{F}, \mu)$ on a differentiable surface F^2 consists of a foliation of F with isolated singularities so that the foliation at the singularities has k-pronged singularities, and a translation-invariant measure μ supported on arcs transverse to the foliation $\mathscr{F}$. We will be interested in foliations all of whose leaves are closed. We will say that two measured foliations are Whitehead equivalent if the foliations can be forced to agree by a series of Whitehead moves and measure preserving isotopies of the underlying surface (see [FLP]).

A holomorphic quadratic differential Φ on a Riemann surface $\mathscr{R}$ is a tensor given locally by an expression $\Phi = q(z)dz^2$, where z is a conformal coordinate on $\mathscr{R}$ and $q(z)$ is holomorphic. Such a quadratic differential Φ defines a measured foliation in the following way. The zeros $\Phi^{-1}(0)$ of Φ are well-defined; away from these zeros, we can choose a canonical conformal coordinate $\zeta = \int^z \sqrt{\Phi}$ so that $\Phi = d\zeta^2$. The local measured foliations $(\{\mathrm{Re}\zeta = \mathrm{const}\}, d(\mathrm{Re}\zeta))$ then piece together to form a measured foliation known as the vertical measured foliation of Φ. There is a corresponding horizontal measured foliation constructed out of the local foliations $(\{\mathrm{Im}\zeta = \mathrm{const}\}, d\,\mathrm{Im}\zeta)$.

Relative to the vertical foliation of a holomorphic quadratic differential, the Riemann surface decomposes into two types of dense open domains: cylindrical domains, foliated by closed curves all freely homotopic to the same closed curve, and spiral domains, in which all leaves are non-compact and dense (see [Str3] and [Gar] for details). We will be interested in quadratic differentials, and more generally, measured foliations, whose foliations consist entirely of cylindrical domains: these are uniquely determined (up to equivalence, in the measured foliation case) by the free homotopy classes of the core curves and the heights of the cylinders ([Str3; § 20]).

2.2. Energy-minimizing maps. Given a map $w\colon \mathscr{R} \to (T, h)$ from a Riemann surface $\mathscr{R}$ to a locally finite Riemannian complex T, we define the energy form to be the tensor $e dz \otimes d\bar{z} = (\|w_* \partial_z\|_h^2 + \|w_* \partial_{\bar{z}}\|_h^2) dz \otimes d\bar{z}$; the energy of the map is $E = \int e dz \wedge d\bar{z}$. An energy minimizing map is a minimum for this functional in a homotopy class. We define the Hopf differential Φ for a map $w\colon \mathscr{R} \to T$ by $\Phi = \Phi dz^2 = 4\langle w_*\partial_z, w_*\partial_z\rangle_h dz^2$. Note that $\|\Phi\| = \|\Phi\|_{L^1} < 2E$.

R. Schoen has emphasized [S] that a map for which the energy functional is stationary under reparametrizations of the domain has a Hopf differential which is holomorphic: one uses suitable domain reparametrizations to show that the Hopf differential satisfies the Cauchy–Riemann equations weakly, and then Weyl's lemma forces the Hopf differential to be (strongly) holomorphic. We observe that in this argument, the range manifold may be singular.

The vertical and horizontal foliations of the Hopf differential for $w\colon \mathscr{R} \to T$ integrate the directions of minimal and maximal stretch of the gradient map dw, for smooth energy minimizing maps $w\colon \mathscr{R} \to T$.

3. Main result

Let $\mathscr{R}$ be a Riemann surface and choose k mutually disjoint homotopically non-trivial Jordan curves $\gamma_1, \dots, \gamma_k$ on $\mathscr{R}$ no pair of which are freely homotopic (an admissable system; see [Str3; §2.6]). Also choose k positive numbers $h_1, \dots, h_k$ to serve as heights, and construct the measured foliation $(\mathscr{F}, \mu)$ on $\mathscr{R}$ consisting entirely of cylinders C_j of height h_j with core curves γ_j and compact singular curves. The leaf space T of $\mathscr{F}$ is a compact 1-complex, possibly with boundary and/or a finite number of finite valence vertices corresponding to the singular curves (see Figures 1 and 2); T inherits a metric $d = d_T = \pi_*\mu$ by pushing forward the measure μ by the natural projection $\pi : \mathscr{R} \to T$ along the leaves. We reprove

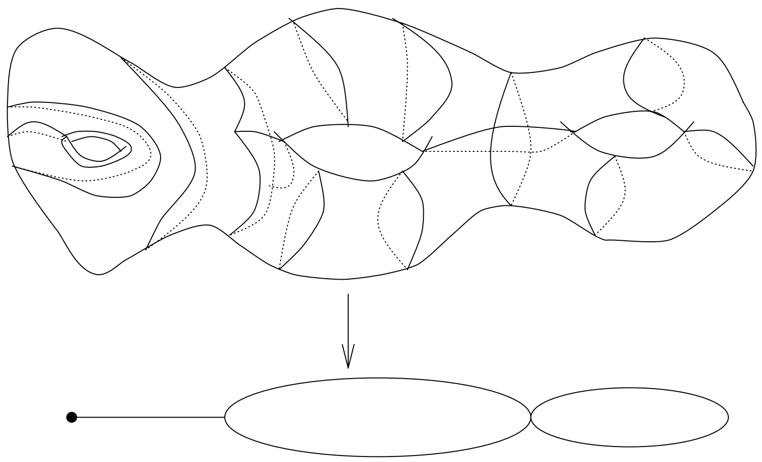

Figure 2.

Theorem 1 ([Ren], [HM]). *There is a holomorphic quadratic differential Φ on $\mathscr{R}$ whose vertical foliation is Whitehead equivalent to $(\mathscr{F}, \mu)$.*

Remark. The novelty of our proof is that Φ will emerge as the Hopf differential of the (unique) energy minimizing map $f\colon \mathscr{R} \to (T, d)$.

Proof of Theorem 1. We first consider the case when $\partial T = \emptyset$. Then by Theorem 2, there is a continuous energy minimizing map $f = f_\pi\colon \mathscr{R} \to T$ homotopic to $\pi\colon \mathscr{R} \to T$. The Hopf differential $\Phi = (f^*d)^{2,0}$ is then holomorphic, and does not vanish everywhere, because this would imply that locally f was either a continuous conformal map between a Riemann surface and a graph, which is absurd, or constant, which is excluded by the homotopy between f and π. We consider its vertical measured foliation. Away from $\Phi^{-1}(0)$, we consider the natural coordinate $\zeta = \int^p \sqrt{\Phi}$ and we write $\zeta = \xi + iy$. Of course, in these coordinates we have $\Phi = 1 d\zeta^2$ so that from our definition of the Hopf differential, we have the two real equations, $\|f_*\partial_\xi\|_d^2 - \|f_*\partial_\eta\|^2 = 1$ and $\langle f_*\partial_\xi, f_*\partial_\eta\rangle_d = 0$, at least away from those neighborhoods $\mathscr{U}$ in $\mathscr{R}$ which are mapped onto three or higher pronged neighborhoods of the singularities of T. Since away from the singularities of T, the graph T is a 1-manifold, we can compare those two equations to find, on those neighborhoods $\mathscr{U} \subset \mathscr{R}$, that $f_*\partial_\xi \| f_*\partial_\eta$. We conclude that $f_*\partial_\eta = 0$ and f_* maps the vector ∂_ξ onto a unit vector tangent to T; note from the uniform continuity of f, that in fact this argument applies to neighborhoods of all but the finitely many arcs in $\mathscr{R}$ representing preimages of a singularity of T. Thus, the map f is in fact a projection along the leaves of the vertical foliation of Φ. Now, a leaf of a vertical foliation of a quadratic differential is either compact and contains a point of $\Phi^{-1}(0)$, or is closed and avoids $\Phi^{-1}(0)$, or is dense in some open set in $\mathscr{R}$ (see [Str3]). If the latter occured, this would force f to be constant on some open set in $\mathscr{R}$; this again forces the holomorphic tensor Φ to vanish on $\mathscr{R}$, a contradiction. The classification of trajectories of a holomorphic quadratic differential on $\mathscr{R}$ then implies that the vertical foliation of Φ is composed of ℓ cylindrical domains, bounded by some compact trajectories emanating from the (finite) set $\Phi^{-1}(0)$.

Recall the set $\mathscr{S}$ of simple closed curves on $\mathscr{R}$. Now $f\colon \mathscr{R} \to T$ is homotopic to $\pi\colon \mathscr{R} \to T$, so the set $A_f = \{[\gamma] \in \mathscr{S} \mid \inf_{\gamma \in [\gamma]} \ell_{d_T}(f(\gamma)) = 0\}$, of curves with representatives whose image under f can have arbitrarily small d_T-length, agrees with A_π. Within $A_f = A_\pi$, we can distinguish $B_f = B_\pi = \{[\gamma] \in A_f \mid i([\gamma], [\beta]) = 0 \ \forall [\beta] \in A_f\}$.

Lemma. $B_\pi = \{[\gamma_1], \dots, [\gamma_k]\}$, *the set of core curves of the foliation* $\mathscr{F}$.

Proof of Lemma. It is clear that $\{[\gamma_1], \dots, [\gamma_k]\} \subset B_\pi$, since each γ_j is clearly in A_π by construction, and any element $[\beta] \in \mathscr{S}$ having an essential intersection with $[\gamma_j]$ would be mapped by π onto a portion of T which would necessarily include the segment of d_T-length h_j corresponding to the projection of the cylinder C_j.

On the other hand, the complement of $\{[\gamma_1], \dots, [\gamma_k]\}$ in A_π consists of classes of curves representable by curves in a singular trajectory of $\mathscr{F}$. Consider such a curve class $[\beta]$ and consider a representative β of $[\beta]$ in a small neighborhood N of the singular trajectory. Since $[\beta] \neq [\gamma_j]$ for any j, we may take $[\beta]$ to be non-boundary parallel. But then there exists a class $[\alpha]$ of simple closed curves in N with $i([\alpha],[\beta]) \neq 0$. (This follows, once we consider a representative β of $[\beta]$ in N: the open manifold N is planar, so β must separate N. But, since β is not boundary parallel, neither component of the complement of β in N is a cylinder, so we can find simple arcs in each component which are homotopically non-trivial, rel β. We then connect these arcs to get the desired curve.) Naturally, then, the class $[\alpha]$ has a representative α in the singular trajectory, so that $\ell_{d_T}(f(\alpha)) = 0$, and so $[\alpha]$, thought of as an element of $\mathscr{S}$, has $[\alpha] \in A_\pi$. We conclude that $[\beta] \in A_\pi - B_\pi$, proving the lemma. □

Conclusion of the proof of Theorem 1. The lemma, applied to both f and π, shows that the vertical foliation for Φ and $\mathscr{F}$ share the same core cylinders. The heights $\{h_j\}$ of the cylinders can be determined by computing $\inf_{\alpha\in[\alpha]} \ell_{d_T}\big(f(\alpha)\big)$ $= \inf_{\alpha\in[\alpha]} \ell_{d_T}\big(\pi(\alpha)\big) = L([\alpha])$ for a sufficient number of curve classes $[\alpha] \in \mathscr{S}$, and solving for heights using the identity $L[\alpha] = \sum_j i([\alpha],[\gamma_j])h_j$.

Since the vertical foliation of Φ and the foliation $\mathscr{F}$ have the same core curves and heights, they are Whitehead equivalent as desired.

In the case where the graph T has boundary $\{p_1,\dots,p_\ell\}$ we proceed as follows. The pre-image of a boundary point consists of a singular trajectory. We consider, for each such boundary point p_j, a simple closed curve α_j contained in the corresponding singular trajectory (the trajectories are required to be homotopically non-trivial), and a vector $\vec{\varepsilon}$ of small numbers $\vec{\varepsilon} = (\varepsilon_1,\dots,\varepsilon_l)$. Our construction gives that the set of curves $\Gamma = \{[\gamma_1],\dots,[\gamma_k],\ [\alpha_1],\dots,[\alpha_\ell]\}$ have mutually disjoint representatives, no pair of which are freely homotopic. We construct the foliation $\mathscr{F}_{\vec{\varepsilon}}$ with core curves from Γ and corresponding heights $\{h_1,\dots,h_k,\ \varepsilon_1,\dots,\varepsilon_\ell\}$; we have chosen the curves α_j so that the corresponding cylinders C'_j will have both boundary components on the boundary of a cylinder C corresponding to one of the γ_i, in fact to the γ_i whose cylinder projected under π to a neighborhood of the boundary point p_j. We see that the graph of the leaf space is boundaryless, so that our previous argument yields a quadratic differential $\Phi_{\vec{\varepsilon}}$ with core curves $\{[\gamma_1],\dots,[\gamma_k],\ [\alpha_1],\dots,[\alpha_\ell]\}$ and heights $\{h_1,\dots,h_k,$ $\varepsilon_1,\dots,\varepsilon_\ell\}$.

We claim that $\|\Phi_{\vec{\varepsilon}}\|$ is uniformly bounded: then in that case, as we let $\vec{\varepsilon} = (\varepsilon_1,\dots,\varepsilon_\ell) \to 0$, the differentials $\Phi_{\vec{\varepsilon}}$ converge uniformly to a quadratic differential with the prescribed core curves and heights.

To see that $\|\Phi_{\vec{\varepsilon}}\|$ is uniformly bounded, we return to the problem of finding an energy minimizing map $w_{\vec{\varepsilon}}\colon \mathscr{R} \to (T_{\vec{\varepsilon}}, d_{\vec{\varepsilon}})$ where $(T_{\vec{\varepsilon}}, d_{\vec{\varepsilon}})$ is the metric graph obtained from the leaf space of the measured foliation $(\mathscr{F}_{\vec{\varepsilon}}, \mu_{\vec{\varepsilon}})$. We construct a competitor $w\colon \mathscr{R} \to (T_{\vec{\varepsilon}}, d_{\vec{\varepsilon}})$ to the energy minimizing map $w_{\vec{\varepsilon}}$ by fixing a vector $\vec{\kappa} = (\kappa_1,\dots,\kappa_l)$ first and considering the energy minimizing map $w_{\vec{\kappa}}\colon \mathscr{R} \to (T_{\vec{\kappa}}, d_{\vec{\kappa}})$. Now, we have constructed $T_{\vec{\kappa}}$ and $T_{\vec{\varepsilon}}$ to be homeomorphic, with a canonical map between the vertices of the graph. We construct an affine map $A_{\vec{\kappa},\vec{\varepsilon}}\colon (T_{\vec{\kappa}}, d_{\vec{\kappa}}) \to (T_{\vec{\varepsilon}}, d_{\vec{\varepsilon}})$ given by extending that map of vertices of the graph to the one-complex in an affine way. The composed map $A_{\vec{\kappa},\vec{\varepsilon}} \circ w_{\vec{\varepsilon}}\colon \mathscr{R} \to (T_{\vec{\varepsilon}}, d_{\vec{\varepsilon}})$ has total energy $E_{\vec{\kappa}}$ exceeding $E(w_{\vec{\varepsilon}})$ because $w_{\vec{\varepsilon}}$ is energy minimizing; moreover, since for $\vec{\varepsilon}$ with $\varepsilon_i < \kappa_i$ for $i = 1, \dots, l$, the map $A_{\vec{\kappa},\vec{\varepsilon}}$ differs from an isometry only by being a contraction on the branches of $T_{\vec{\kappa}}$ dual to the core curves $\{[\alpha_1], \dots, [\alpha_l]\}$, we see that we have the uniform bound $E_{\vec{\kappa}} < E_0$. This yields the estimate $\|\Phi_{\vec{\varepsilon}}\| < 2E(w_{\vec{\varepsilon}}) < 2E_{\vec{\kappa}} < 2E_0$ (where we recall the first inequality from §2), proving the claim.

4. Appendix

We prove, with an argument (see [J1; §4.1], and [GS]) that is already almost entirely in the literature,

Theorem 2. *Let $\phi\colon \mathscr{R} \to (T, d)$ be a map from a Riemann surface $\mathscr{R}$ to a graph T, where T is equipped with a metric d; suppose $\phi \in C^0 \cap H^1(\mathscr{R}, T)$. Then there is a continuous energy minimizing map $u\colon \mathscr{R} \to (T, d)$ homotopic to ϕ with the modulus of continuity of u estimable in terms of $E(\phi)$ and the modulus of continuity of ϕ.*

Remark. We include this proof for the purpose of displaying the analytical underpinnings of our approach to Strebel's theorem, not for any claim of novelty.

Proof. The plan is to prove the result first locally, and then to piece together the local harmonic maps into a well controlled sequence of maps which tend towards a minimizer $u\colon \mathscr{R} \to (T, d)$.

4.1. So consider first a continuous map of the circle $g\colon \partial\Delta \to (T^*, d)$ where g admits an extension $\bar{g}\colon \partial\Delta \to (T^*, d)$ of finite energy and the image of g is a simply connected closed subgraph $T^* \subset T$, for instance the intersection of T with a ball, $T^* = T \cap B(p, r)$. We claim that there exists a harmonic map $h\colon \Delta \to (T, d)$ with boundary values g, and that h minimizes the energy with respect to these boundary values. We also claim that the modulus of continuity of h can be estimated in terms of $E(\bar{g})$ and the modulus of continuity of g. To see this, take a minimizing sequence $\{v_i\}$ for the energy in $V = \{v \in H^1(\Delta, B(p, r^*)),\ v\,|_{\partial\Delta} = g\}$ where $r < r^*$. (The point here is that our space V makes perfectly good sense for a singular target being a graph, since we can embed the simply connected graph T^* isometrically in R^2, defining $H^1(\Delta, B(p, r^*)) = \{v \in H^1(\Delta, \mathbf{R}^2);\ v(z) \in B(p, r^*) \subset T^* \subset \mathbf{R}^2$ a.e. $z \in \Delta\}$ and $v = \hat{v}$ on $\partial\Delta$ if $v - \hat{v} \in H^1_0(\Delta, \mathbf{R}^2)$ where $H^1_0(\Delta, \mathbf{R}^2)$ is the H^1-norm closure of smooth, compactly supported $\mathbf{R}^2$-valued maps on Δ.) Now, such a minimizing sequence $\{v_i\}$ has a subsequence converging weakly in H^1, and the limit h minimizes energy in its class because of the lower semicontinuity of the energy functional. Since the set $\{v \in H^1(\Delta, \mathbf{R}^2),\ v\,|_{\partial\Delta} = g\,|_{\partial\Delta}\}$ is a closed affine subspace of $H^1(\Delta, \mathbf{R}^2)$, it is weakly closed, and we conclude that $h\,\big|_{\partial\Delta} = g\,\big|_{\partial\Delta}$. We can estimate the modulus of continuity of h as follows: the Courant–Lebesgue lemma ([J1; Lemma 3.1.1]) provides that for each $x \in \mathscr{R}$ and $\varepsilon > 0$ there is a ϱ depending only upon ε, $E(\bar{g})$ and the modulus of continuity of g and $q \in T^*$ so that $h(\partial B(x, \varrho)) \subset B(q, \varepsilon) \cap T^*$. But then, since $B(q, \varepsilon) \cap T^*$ is convex, the maximum principle forces $h(B(x, \varrho)) \subset B(q, \varepsilon) \cap T^*$. Thus we have shown the continuity of h and estimated its modulus of continuity.

Remark. Of course, when $T^* \subset T$ is an interval, h is given classically by the Poisson integral formula, so here we are really only interested in T^* being a non-trivial graph, for instance a "Y". It would be interesting to have a more explicit description of the map $h\colon \overline{\Delta} \to T^*$ in this case, in terms of the boundary values $h\,\big|_{\partial\Delta}\colon \partial\Delta \to T^*$.

4.2. To complete the argument, we choose a δ_0 sufficiently small so that the Courant–Lebesgue lemma forces a harmonic map ϕ with energy $E(\phi)$ of the ball of radius $\delta < \delta_0$ to be contained in a simply connected subgraph $T^* \subset T$; we want to be able to apply the construction of the previous paragraph 4.1.

Choose $\delta < \delta_0$, and cover $\mathscr{R}$ by a finite number M of balls $B(x_i, \delta/2)$, $i = 1, \dots, M$; here we have chosen some suitable background metric on $\mathscr{R}$. Following [J1; Theorem 4.1.1], we let u_n be a continuous energy minimizing sequence of maps homotopic to ϕ; we may as well assume that $E(u_n) < E(\phi)$ so that we have control on $u_n(B(x_i, \delta/2))$. Thus, by the Courant–Lebesgue lemma and our choice of constants, for every n, we can find $r_{n,1}$, with $\delta < r_{n,1} < \delta^{1/2}$ and $p_{n,1} \in N$ so that $u_n(\partial B(x_1, r_{n,1})) \subset B(p_{n,1}, r^*)$ for r^* so small that $B(p, r^*) \subset T$ is always simply connected. Thus, we can invoke our construction for harmonic maps of balls into simply connected graphs: we replace $u_n\,\big|_{B(x_1, r_{n,1})}$ by the harmonic map $h_{x_1,n}\colon B(x_1, r_{n,1}) \to B(p_{n,1}, r)$ whose boundary values $h_{x_1,n}\,\big|_{\partial B(x_1, r_{n,1})}$ agree pointwise with those $u_n\,\big|_{\partial B(x_1, r_{n,1})}$ of u_n.

We can assume that $r_{n,1} \to r_1$ and using the estimates on the modulus of continuity of the harmonic maps $h_{x_1,n}$ of simply connected domains, we can take the replaced maps, say $u_{n,1}$, to converge uniformly on $B(x_1, \delta - \eta)$ for any $0 < \eta < \delta$. Naturally $E(u_{n,1}) < E(u_n)$.

We repeat the argument to find a radius $r_{n,2}$ with $\delta < r_{n,2} < \sqrt{\delta}$ and $u_{n,1}(\partial B(x_2, r_{n,2})) \subset B(p_{n,2}, r)$ and replace $u_{n,1}$ on $B(x_2, r_{n,2})$ by the appropriate solution to the Dirichlet problem; we denote the new replaced maps by $u_{n,2}$, and again assume that $r_{n,2} \to r_2$.

Now, in the first replacement step, $u_{n,1}$ converged uniformly on $B(x_2, r_2) \cap B(x_1, \delta - \eta/2)$, and thus the boundary values for our second replacement step converge uniformly on $\partial B(x_2, r_{n,2}) \cap B(x_1, \delta - \eta/2)$. Of course, we have a uniform estimate for the modulus of continuity between small disks and portions of graphs (even after allowing for the difference between our background metric and the Euclidean metric on the disk): this means that we can assume that the maps $u_{n,2}$

converge uniformly on $B(x_1, \delta - \eta) \cup B(x_2, \delta - \eta)$ for $0 < \eta < \delta$. Of course, the act of replacing lowers energy, and we conclude that $E(u_{n,2}) \le E(u_{n,1}) \le E(u_n)$. We repeat the replacement argument in this way obtaining a family of maps $u_{n,M}$ with $E(u_{n,M}) \le E(u_n)$, and which converge uniformly on all balls $B(x_i, \delta/2)$, hence on $\mathscr{R}$ since $\mathscr{R} \subset \bigcup_{i=1}^{M} B(x_i, \delta/2)$.

We let u denote the limit of $u_{n,M}$ as $n \to \infty$ (recall that M is a fixed number depending only upon δ). Note that since replacement on disks will not affect the homotopy class of the maps $u_{n,i}$, the uniform convergence of $u_{n,M}$ to u forces u to be not only continuous but also homotopic to the given $\phi\colon \mathscr{R} \to T$. Naturally also, since u_n is a minimizing sequence, so is $u_{n,M}$; we note that since $E(u_{n,M}) \le E(\phi)$, the maps $u_{n,M}$ converge weakly in $H^1(\mathscr{R}, T)$ (see above comments on proper definitions), and by the lower semicontinuity of the energy functional with respect to weak H^1 convergence, we get that the limit u of the energy minimizing sequence u_n^M minimizes energy within the homotopy class of $\phi\colon \mathscr{R} \to T$.

Remarks. 1) When the image of u is contained in an embedded interval in the graph, it follows immediately that u is a harmonic function, and hence real analytic. 2) It is possible to start with $\phi\colon \mathscr{R} \to T$ being a projection as in the application (Theorem 1), and then argue that all replacements involve only Whitehead moves to the resulting Hopf differential vertical foliation. This yields the desired Jenkins–Strebel differential possibly more constructively.

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