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PATTERSON MEASURE AND UBIQUITY

M.M. Dodson, M.V. Melián, D. Pestana, and S.L. Velani*

University of York, Department of Mathematics, York YO1 5DD, England
Universidad Autónoma de Madrid, Departamento de Matemáticas, 28049 Madrid, Spain
Universidad Autónoma de Madrid, Departamento de Matemáticas, 28049 Madrid, Spain
University of York, Department of Mathematics, York YO1 5DD, England

Abstract. Let L be a closed subset of $\mathbf{R}^k$, with Hausdorff dimension δ, which supports a probability measure m for which the m-measure of a ball of radius r and centred at a point in L is comparable to r^δ. By extending the notion of ubiquity from k-dimensional Lebesgue measure to m, a natural lower bound for the Hausdorff dimension of a fairly general class of lim sup subsets of L is obtained. This is applied to Patterson measure supported on the limit set of a convex co-compact group, to obtain the Hausdorff dimension of the set of 'well-approximable' points associated with the limit set. The equivalent geometric result in terms of geodesic excursions on the quotient manifold is also obtained. These results are counterparts of Jarník's theorem on simultaneous diophantine approximation.

1. Introduction

The idea of ubiquity [4] has been used to obtain a lower bound for the Hausdorff dimension of a wide range of sets which arise in the theory of metric Diophantine approximation [3], [4], [5]. In these applications, which include the Jarník–Besicovitch theorem [2] [7], the sets in question are those of 'well-approximable' points in either Euclidean space or submanifolds of Euclidean space. Although of a fairly general geometric and statistical character, the original formulation of ubiquity used open or relatively open 'approximating' sets.

Recently, ubiquity has been used to establish the natural analogue of the Jarník–Besicovitch theorem for non-elementary geometrically finite groups of the first kind [15], [16]. These analogues have also been obtained by utilising the notion of a 'well distributed system' [8]. This is a generalisation of the concept of a regular system introduced by A. Baker and W.M. Schmidt [1]. In this setting it is essentially equivalent to the restricted definition of ubiquity given in [15].

For an arbitrary geometrically finite group, the upper bound for the Hausdorff dimension of the set of 'well-approximable' points presents no difficulty, as is commonly the case. In the case of groups of the first kind, the limit set is the unit sphere S^k and therefore of positive k-dimensional Lebesgue measure. This allows

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* Present address: Mathematisches Institut, Bunsenstraße 3–5, D-37073 Göttingen, Germany.

the establishment of ubiquity and hence the complementary lower bound. For geometrically finite groups of the second kind, the limit set is of zero k-dimensional Lebesgue measure in S^k. Consequently without modification, ubiquity cannot be applied to obtain the analogue of the Jarník–Besicovitch theorem for groups of the second kind.

In this paper, ubiquity and its consequences are extended to measures satisfying the properties of *Patterson measure* [10] in the case of non-elementary geometrically finite groups without parabolic elements, i.e. convex co-compact groups. This extension is not routine and requires new ideas, although the basic structure of the proof of Theorem 1 (Section 3) follows that of Theorem 1 in [4]. For convex co-compact groups, Patterson measure is comparable to Hausdorff measure and is easily seen to be the appropriate measure for the problem of metric diophantine approximation, see Theorem 3 (Section 4). However, for groups of the second kind with parabolic elements, Patterson measure is no longer comparable to Hausdorff measure and the analysis is more difficult.

The proof of Theorem 3 can also be obtained by extending the concept of a 'well distributed system', already mentioned. This involves the construction of a 'Cantor-like' subset of the lim-sup set appearing in Theorem 3, on which a probability measure satisfying the mass distribution principle [6] is constructed. However, the ubiquity formulation allows one to obtain a more general result concerning the Hausdorff dimension of a fairly general class of lim-sup sets. Furthermore, under certain circumstances ubiquity allows one to obtain the Hausdorff measure at the critical exponent.

The paper is organised as follows. In Section 2, the required properties of a measure m are defined and Hausdorff measure and dimension are also defined in a manner appropriate for the setting of this paper. In Section 3, ubiquity is extended to the measures m introduced in the previous section, i.e. to m-ubiquity. Consequently, results are obtained for the Hausdorff dimension of lim-sup sets whose underlying sets are of positive m-measure. Finally, in Section 4, m-ubiquity is used to obtain the analogue of the Jarník–Besicovitch theorem for convex co-compact groups. Furthermore, the equivalent geometric result analogous to Theorem 1 in [8] is outlined.

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2. Centred measure and dimension

Throughout this paper, L will be a closed non-empty subset of $\mathbf{R}^k$ on which a non-atomic probability measure m is supported. Suppose further that there exists a fixed positive $\delta \le k$, and positive constants a, b, r_0 such that for each Euclidean ball $B(c,r)$ in $\mathbf{R}^k$ centred at $c \in L$ with radius $r \le r_0$,

$$ar^\delta < m\big(B(c,r)\big) < br^\delta. \tag{1}$$

To simplify notation the symbols $\ll$ and $\gg$ will be used to indicate an inequality with an unspecified positive multiplicative constant. If $a \ll b$ and $a \gg b$ we write $a \sim b$, and say that the quantities a and b are comparable. Thus (1) can be written as $m\big(B(c,r)\big) \sim r^\delta$ and any small Euclidean ball centred at a point of L has m-measure comparable to r^δ. In particular, when $\delta = k$, the measure m is comparable to the k-dimensional Lebesgue measure. The conditions on the measure m are chosen with Patterson measure on a limit set (see Section 4) in mind.

When there is no risk of ambiguity, we will write $B = B(c,r)$. For any positive θ, define θB to be the ball $B(c,\theta r)$ of centre c and radius θr. Also $r(B)$ will denote the radius of the ball B.

Now let $\mathscr{B}$ be a finite or countable collection of balls $B \subset \mathbf{R}^k$ with centres c in L and radii $r(B)$. Let F be any non-empty subset of L of $\mathbf{R}^k$. If $F \subset \bigcup_{B\in\mathscr{B}} B$ and $0 < r(B) \leq \varrho$ for each B in $\mathscr{B}$, then the collection of balls $\mathscr{B}$ is said to be an *L-centred ϱ-cover* of F.

Let s be a positive number and for any positive ϱ define

$$\mathscr{H}^s_{\varrho;L}(F) = \inf\Big\{ \sum_{B\in\mathscr{B}} r(B)^s : \mathscr{B} \text{ is an } L\text{-centred } \varrho\text{-cover of } F \Big\}$$

where the infimum is over all (countable) L-centred ϱ-covers of F. The *L-centred s-dimensional Hausdorff measure* $\mathscr{H}^s_L(F)$ of F is defined by

$$\mathscr{H}^s_L(F) = \lim_{\varrho\to 0} \mathscr{H}^s_{\varrho;L}(F) = \sup_{\varrho>0} \mathscr{H}^s_{\varrho;L}(F)$$

and the *L-centred Hausdorff dimension* $\dim_L F$ of F by

$$\dim_L F = \inf\{s : \mathscr{H}^s_L(F) = 0\} = \sup\{s : \mathscr{H}^s_L(F) = \infty\}.$$

With $L = \mathbf{R}^k$, the definitions of L-centred s-dimensional Hausdorff measure $\mathscr{H}^s_L(F)$, and L-centred Hausdorff dimension $\dim_L F$, are the usual s-dimensional Hausdorff measure $\mathscr{H}^s(F)$, and Hausdorff dimension $\dim F$ respectively. Further details and alternative definitions of Hausdorff measure and dimension can be found in [6].

Lemma 1. *For any subset F of L*

$$\mathscr{H}^s(F) \sim \mathscr{H}^s_L(F).$$

Proof. Since the covers used in $\mathscr{H}^s_L$ are restricted to L-centred balls,

$$\mathscr{H}^s_L(F) \geq \mathscr{H}^s(F).$$

In the other direction, any ϱ-cover of F by balls C_i (which without loss of generality all meet F) can be replaced by an L-centred cover of balls $\widehat{C}_i$ of radius $2r_i$. It follows that for any ϱ with $0 < \varrho < r_0/2$,

$$\mathscr{H}^s_{\varrho,L}(F) \leq \mathscr{H}^s_\varrho(F),$$

whence letting $\varrho \to 0$,

$$\mathscr{H}^s_L(F) \ll \mathscr{H}^s(F),$$

as required.

It follows, as a direct consequence of Lemma 1 and the definitions of Hausdorff dimension and L-centred Hausdorff dimension, that

$$\dim F = \dim_L F.$$

Remark 1. The inequality (1) implies that

$$\limsup_{r\to 0} \frac{m\big(B(c,r)\big)}{r^\delta} \sim 1,$$

for all closed Euclidean balls $B(c,r)$ with centre c in L and radii r. It follows from the density theorem in [6, p. 61] that for L compact,

$$\dim L = \delta.$$

In fact, it can be deduced from inequality (1) that the measure m is comparable to the δ-dimensional Hausdorff measure on L. Another consequence of the density theorem is that for any subset F of L with positive m-measure, $\dim F = \delta$. In general the converse is not true.

In the application considered in this paper, δ will indeed correspond to the Hausdorff dimension of the set L, i.e. the Hausdorff dimension of the ambient space of any subset of L.

3. m-ubiquitous systems

Let U be an open ball with radius $r(U)$ centred on a point $c(U)$ of L and let

$$\Omega = U \cap L.$$

Consider the generalized lim-sup set of the form

$$\Lambda_{\mathscr{F}} = \{\mathbf{x} \in \Omega : \mathbf{x} \in F_\alpha \text{ for infinitely many } \alpha \text{ in } J\},$$

where $\mathscr{F} = \{F_\alpha : \alpha \in J\}$ is a family of subsets of Ω indexed by a countable set J. When J is the set of positive integers, $\Lambda_{\mathscr{F}}$ is the familiar lim-sup of the sequence of sets F_j, $j = 1, 2, \ldots$, i.e. $\Lambda_{\mathscr{F}} = \bigcap_{N=1}^{\infty} \bigcup_{j=N}^{\infty} F_j$. Ubiquitous systems were introduced in [4] and used to obtain a lower bound for the Hausdorff dimension of sets of the form $\Lambda_{\mathscr{F}}$ where the sets F_α in $\mathscr{F}$ are (or contain) open neighbourhoods of certain sets R_α, called *resonant sets* (see [4, Theorem 1]). However in [4], the set Ω is a non-empty open bounded subset of $\mathbf{R}^k$ and hence has positive Lebesgue measure. The setting of ubiquity is now extended to a more general measure m, although because of the particular application to convex co-compact groups the resonant sets R_α will be taken to be points. Using this m-ubiquity, an analogue of Theorem 1 in [4] is obtained for a certain class of subsets Ω of L which are of zero Lebesgue measure but of positive m-measure.

Definition 1. Let $\mathscr{R} = \{R_\alpha \in \Omega : \alpha \in J\}$ be a set of points in Ω and let $\beta\colon J \to \mathbf{R}^+\colon \alpha \mapsto \beta_\alpha$ be a positive function on J and for each N in $\mathbf{N}$ define $J(N) = \{\alpha \in J : \beta_\alpha \leq N\}$. Suppose that there exists a positive decreasing function $\lambda\colon \mathbf{N} \to \mathbf{R}^+$ with $\lambda(N) \to 0$ as $N \to \infty$, such that

$$(2) \qquad m\Bigl(\Omega \backslash \bigcup_{\alpha \in J(N)} \tfrac{1}{3} B\bigl(R_\alpha, \lambda(N)\bigr)\Bigr) \to 0$$

as $N \to \infty$. Then the pair $(\mathscr{R}, \beta)$ is said to be an m*-ubiquitous* system relative to λ.

It follows from the definition that for each positive integer N there exists an m-measurable subset $A(N)$ of Ω, namely $\bigcup_{\alpha \in J(N)} \frac{1}{3} B\bigl(R_\alpha, \lambda(N)\bigr)$ in the above definition, and a positive number $\lambda(N)$ such that:

(i) for any ball B in U centred on a point of Ω with $r(B) = \lambda(N)$ and $\frac{1}{2} B \cap A(N) \neq \emptyset$, there exists an α in $J(N)$ such that for all ϱ satisfying $0 < \varrho \leq \lambda(N)$,

$$(3) \qquad m\bigl(B \cap B(R_\alpha, \varrho)\bigr) \sim \varrho^\delta$$

for N sufficiently large; where $0 < \delta \leq k$ is the fixed constant associated with the measure m. The implied constants in (3) are dependent only on Ω and the pair $(\mathscr{R}, \beta)$; and

(ii)

$$(4) \qquad \lim_{N \to \infty} m\bigl(\Omega \backslash A(N)\bigr) = 0.$$

The conditions (3) and (4) are the key to deriving a lower bound for the Hausdorff dimension of lim-sup sets of the form Λ_F, and so we will work with them. In fact the more general definition of ubiquity in the Lebesgue measure setting given in [4] is in terms of conditions corresponding to (3) and (4). The set $A(N)$ is an approximating set for Ω in the measure theoretical sense and is not required to be a union of open balls centred at 'special' resonant points as in Definition 1 above. The factor of $\frac{1}{3}$ appearing in (2) is to ensure consistency with the definition of ubiquity given in [15].

The essential feature of a m-ubiquitous system $(\mathscr{R}, \beta)$ is that for each positive integer N and any $\mathbf{x}$ in $A(N)$, there exists an α in $J(N)$ such that the inequality $|\mathbf{x}-R_\alpha| < 2\lambda(N)$ is satisfied. This is guaranteed by condition (3) above. Condition (4) implies that the m-measure of the set of points in Ω not lying within a distance $2\lambda(N)$ of a resonant point R_α with α in $J(N)$, tends to zero as N tends to infinity. With $A(N) = \bigcup_{\alpha\in J(N)} \frac{1}{3}B(R_\alpha, \lambda(N))$ as in the definition, both these features are immediate from (2). In applications the function λ associated with a system $(\mathscr{R}, \beta)$ arises naturally from the theory of Diophantine approximation, such as for example Dirichlet's theorem.

Let ψ: $\mathbf{R}^+ \to \mathbf{R}^+$ be a decreasing function with $\psi(x) \to 0$ as $x \to \infty$. Write

$$\Lambda(\psi) = \big\{\mathbf{x} \in \Omega : |\mathbf{x} - R_\alpha| < \psi(\beta_\alpha) \text{ for infinitely many } \alpha \text{ in } J\big\}.$$

If the pair $(\mathscr{R}, \beta)$ and the function ψ can be chosen such that for each α in J,

$$B\big(R_\alpha, \psi(\beta_\alpha)\big) \subseteq F_\alpha$$

then clearly $\Lambda(\psi) \subseteq \Lambda_F$ and a lower bound for $\dim \Lambda(\psi)$ is also a lower bound for $\dim \Lambda_F$.

In determining a lower bound for $\Lambda(\psi)$ a type of asymptotic density, given by the function λ, of the resonant points R_α in Ω is required. This ensures that 'most' points in Ω are close to some point R_α with the 'size' of the 'denominator' β_α not too large. A lower bound for $\Lambda(\psi)$ is given by the following theorem (see [4] for a more general result in the Lebesgue measure setting).

Theorem 1. *Suppose that $(\mathscr{R}, \beta)$ is an m-ubiquitous system with respect to λ and that ψ: $\mathbf{R}^+ \to \mathbf{R}^+$ is a decreasing function. Then*

$$\dim \Lambda(\psi) \geq \gamma\, \delta$$

where

$$\gamma = \min\left\{1, \limsup_{N\to\infty} \frac{\log \lambda(N)}{\log \psi(N)}\right\}.$$

If there exists a strictly increasing sequence $\{N_r : r = 0, 1, 2, \ldots\}$ such that $\lambda(N_r)/\psi(N_r)^\gamma \to 0$ as $r \to \infty$, then

$$\mathscr{H}^{\delta\,\gamma}\big(\Lambda(\psi)\big) = \infty.$$

The following covering result is used repeatedly throughout the proof of Theorem 1 and is therefore included at this point.

Lemma 2. *Let $\mathscr{B}$ be a collection of balls contained in a bounded subset of $\mathbf{R}^k$. Then there exists a finite or countably infinite disjoint subcollection $\{B_i\}$ such that*

$$\bigcup_{B\in\mathscr{B}} B \subset \bigcup_i \widetilde{B}_i$$

where $\widetilde{B}_i = 5B_i$.

For the proof of Lemma 2 the reader is referred to [6, p. 60].

Proof of Theorem 1. By Lemma 1, $\dim \Lambda(\psi) = \dim_L \Lambda(\psi)$ and so it is sufficient to show that

$$\dim_L \Lambda(\psi) \geq \gamma\delta.$$

Throughout the proof N will be taken large enough to ensure that $\lambda(N)$ is small enough so that

$$m\big(B(c, 2\lambda(N))\big) \sim \lambda(N)^\delta$$

for any ball B centred on a point of L with radius $2\lambda(N)$.

Define

$$U_N^- = \big\{\mathbf{x} \in U : |\mathbf{x} - c(U)| < r(U) - 10\lambda(N)\big\}$$

to be the open ball concentric with U of radius $r(U_N^-) = r(U) - 10\lambda(N)$, and let

$$\Omega_N^- = U_N^- \cap L.$$

Then by construction and the continuity of measures $m(\Omega_N^-) \to m(\Omega)$ as $N \to \infty$, whence $m(\Omega) \sim m(\Omega_N^-)$ for N sufficiently large.

Consider the collection $\mathscr{B}_*(N)$ of open balls with radii $2\lambda(N)$ and centres at each point of Ω_N^-. By Lemma 2, there exists a finite disjoint subcollection $\mathscr{B}(N)$ of $\mathscr{B}_*(N)$ such that

$$m\Big(\Omega_N^- \backslash \bigcup_{B\in\mathscr{B}(N)} \widetilde{B}\Big) = 0$$

where $\widetilde{B} = 5B$. Then for any ball B in $\mathscr{B}(N)$, $B \subset U$ and moreover $\widetilde{B} \subset U$. By construction,

$$m(\Omega) \geq m\Big(\bigcup_{B\in\mathscr{B}(N)} B\Big) = \sum_{B\in\mathscr{B}(N)} m(B) \sim \#\mathscr{B}(N)\lambda(N)^\delta$$

and

$$m(\Omega_N^-) \leq m\Big(\bigcup_{B\in\mathscr{B}(N)} \widetilde{B}\Big) \sim \#\mathscr{B}(N)\lambda(N)^\delta,$$

where $\#A$ denotes the cardinality of the set A. Hence for N sufficiently large,

$$m(\Omega) \sim \#\mathscr{B}(N)\lambda(N)^\delta$$

since $m(\Omega_N^-) \sim m(\Omega)$, and so $\#\mathscr{B}(N) \sim m(\Omega)\lambda(N)^{-\delta}$.

Good sampling sets. Next let $\mathscr{E}(N)$ denote the 'exceptional' set of balls B in $\mathscr{B}(N)$ for which $\frac{1}{4}B \cap A(N) = \emptyset$ and let $\mathscr{G}(N) = \mathscr{B}(N) \backslash \mathscr{E}(N)$ be the complementary set of 'good' balls B in $\mathscr{B}(N)$ for which $\frac{1}{4}B \cap A(N) \neq \emptyset$. It follows that

$$m\big(\mathscr{E}(N)\big) = m\Big(\bigcup_{B \in \mathscr{E}(N)} B\Big) \sim \#\mathscr{E}(N)\lambda(N)^{\delta} \longrightarrow 0$$

as $N \to \infty$, since otherwise the measure of points not in $A(N) \subset \Omega$ would be greater than some positive constant, contradicting the ubiquity condition (4), namely that $m(\Omega \backslash A(N)) \to 0$ as $N \to \infty$. Hence

$$m\big(\mathscr{G}(N)\big) = m\Big(\bigcup_{B \in \mathscr{G}(N)} B\Big) \to m(\Omega)$$

as $N \to \infty$, and so for N sufficiently large $m\big(\mathscr{G}(N)\big) \sim m(\Omega)$. Thus for sufficiently large N,

$$\#\mathscr{G}(N) = \#\mathscr{B}(N) - \#\mathscr{E}(N) \sim \lambda(N)^{-\delta} m(\Omega). \tag{5}$$

For each ball B in $\mathscr{G}(N)$ choose α in $J(N) = \{\alpha \in J : \beta_\alpha \leq N\}$ such that (3) holds for $\frac{1}{2}B$. As a consequence of the ubiquity condition (3), $R_\alpha \in \frac{1}{2}B$. Let $W(B)$ be the set of $\mathbf{x}$ in B such that

$$|\mathbf{x} - R_\alpha| \leq \psi(N) \leq \psi(\beta_\alpha), \tag{6}$$

and let $V(B) = W(B) \cap L$. By construction, $V(B)$ is a closed subset of $B \cap \Omega$. Now let

$$T(N) = \bigcup_{B \in \mathscr{G}(N)} V(B) \quad \text{and} \quad T^\infty = \limsup\big\{T(N) : N \in \mathbf{N}\big\}.$$

Thus $T(N)$ is the union of a finite number of disjoint closed subsets $V(B)$ of Ω and so it too is a closed subset of Ω. Without loss of generality, take $\psi(N) \leq \lambda(N)$ for $N = 1, 2, \ldots$, so that $\log \lambda(N)/\log \psi(N) \leq 1$ for N sufficiently large. It follows that $B\big(R_\alpha, \psi(N)\big) = W(B)$. Then by (3), (5), (6) and the fact that the measure m is supported on L, for all sufficiently large N, $m\big(V(B)\big) = m\big(W(B)\big) \sim \psi(N)^\delta$ and

$$m\big(T(N)\big) = m\Big(\bigcup_{B \in \mathscr{G}(N)} V(B)\Big) \sim \#\mathscr{G}(N)\psi(N)^\delta \sim \Big(\frac{\psi(N)}{\lambda(N)}\Big)^\delta. \tag{7}$$

Now suppose $\mathbf{x} \in T^\infty$. Then by definition, the inequality (6) holds for infinitely many integers N_r with α in $J(N_r)$, $r = 1, 2, \ldots$. If $\mathbf{x} \neq R_\alpha$ for any α

in J, then (6) must hold for infinitely many distinct α in J since $\psi(\mathbf{x}) \to 0$ as $\mathbf{x} \to \infty$ and so $\mathbf{x} \in \Lambda(\psi)$. If $\mathbf{x} = R_\alpha$ for some α in J, inequality (6) may hold for just finitely many distinct α in J, in which case $\mathbf{x} \notin \Lambda(\psi)$. However, since the set J is countable, the set of such points

$$T_1^\infty = \{\mathbf{x} \in T^\infty : \mathbf{x} = R_\alpha \text{ for some } \alpha \in J\}$$

is also countable, so that $\mathscr{H}_L^s(T_1^\infty) = 0$. Hence, the set of points T^∞ can be written as the union of two disjoint sets T_1^∞ and T_2^∞, where T_1^∞ is the countable set above and T_2^∞ is a proper subset of $\Lambda(\psi)$. Thus

$$\mathscr{H}_L^s(T^\infty) = \mathscr{H}_L^s(T_1^\infty) + \mathscr{H}_L^s(T_2^\infty) = \mathscr{H}_L^s(T_2^\infty) \leq \mathscr{H}_L^s\big(\Lambda(\psi)\big),$$

and a lower bound for $\dim_L T^\infty$ is also a lower bound for $\dim_L \Lambda(\psi)$.

The interior of a subset X of L with respect to L will be denoted by $\dot{X}$, and the closure of a subset X of L with respect to L by $\bar{X}$. The interior and closure of a subset X of $\mathbf{R}^k$ with respect to the usual topology of $\mathbf{R}^k$ will be denoted by $\operatorname{int}(X)$ and $\operatorname{cl}(X)$ respectively. It should be noted that for any ball B of radius $r(B) < r_0$ with centre in L, $m\big(\operatorname{int}(B)\big) \sim m\big(\operatorname{cl}(B)\big)$. Hence, for sufficiently large N,

$$m\big(\dot{T}(N)\big) \sim m\big(T(N)\big).$$

Lemma 3. *For any closed set $X \subset \Omega$, there exists an integer $N^*(X)$ such that for all $N \geq N^*(X)$,*

$$m\big(\dot{T}(N) \cap \dot{X}\big) \geq K_1 m\big(\dot{T}(N)\big) m(\dot{X}) \tag{8}$$

where the positive constant K_1 does not depend on X or N.

Proof. Without loss of generality assume that $m(\dot{X}) > 0$, whence $\dot{X} \neq \emptyset$. Let ∂X denote the boundary of the set X with respect to L, and for any positive ϱ write

$$X_\varrho = \big\{\mathbf{x} \in X : \operatorname{dist}_\infty(\mathbf{x}, \partial X) > \varrho\big\},$$

where $\operatorname{dist}_\infty(\mathbf{x}, \partial X) = \inf\big\{|\mathbf{x} - \mathbf{y}| : \mathbf{y} \in \partial X\big\}$. By construction and the continuity of measures $m(X_\varrho) \to m(\dot{X})$ as $\varrho \to 0$, whence for $\varrho_* = \varrho_*(X)$ sufficiently small

$$m(\dot{X}) \leq 2m(X_{\varrho_*}).$$

Let

$$\mathscr{G}_X(N) = \big\{B \in \mathscr{G}(N) : B \cap L \subseteq \dot{X}\big\}.$$

The set L is a closed subset of $\mathbf{R}^k$ by hypothesis and X is a closed bounded subset of L with respect to the relative topology of L, since $X \subset \Omega$. Then there exists a positive $\sigma = \sigma(\varrho_*, X)$ such that any ball B with $r(B) \leq \sigma$ and

$B \cap X_{\varrho *} \neq \emptyset$, satisfies $B \cap L \subseteq \dot{X}$. Thus for sufficiently large $N \geq N_1(X)$ so that $\lambda(N) \leq \sigma/10$, for any ball B in $\mathscr{G}(N)$ with $B \cap X_{\varrho *} \neq \emptyset$ implies that $B \cap L \subseteq \dot{X}$, i.e. $B \in \mathscr{G}_X(N)$. Moreover for $N \geq N_1(X)$, for any ball B in $\mathscr{G}(N)$ with $5B \cap X_{\varrho *} \neq \emptyset$ implies that $5B \cap L \subseteq \dot{X}$, i.e. $5B \in \mathscr{G}_X(N)$.

Also for sufficiently large $N \geq N_2(X)$, $\dot{X} \subseteq \Omega_N^-$, since $X \subset \Omega$ and $\Omega_N^- \to \Omega$ as $N \to \infty$. Hence

$$\dot{X} \subset \bigcup_{B \in \mathscr{B}(N)} \tilde{B}.$$

Moreover, by the construction of the set $X_{\varrho *}$, for $N \geq \max\{N_1(X), N_2(X)\}$,

$$X_{\varrho *} \subseteq \bigcup_{B \in \mathscr{G}_X(N)} \tilde{B} \cup \bigcup_{B \in \mathscr{E}(N)} \tilde{B}.$$

However $m(\mathscr{E}(N)) \sim m(\bigcup_{B \in \mathscr{E}(N)} \widetilde{B}) \to 0$ as $N \to \infty$, whence $m(\bigcup_{B \in \mathscr{E}(N)} \widetilde{B})$ can be made arbitarily small. Hence for $N \geq N^*(X) = \max\{N_1(X), N_2(X)\}$,

$$m(\dot{X}) \leq 2m(X_{\varrho *}) \ll \sum_{B \in \mathscr{G}_X(N)} m(\widetilde{B}) \ll \#\mathscr{G}_X(N)\, \lambda(N)^{\delta},$$

from which it follows that $\#\mathscr{G}_X(N) \gg m(\dot{X})\lambda(N)^{-\delta}$.

Now $\dot{T}(N) \cap \dot{X} \supseteq \bigcup_{B \in \mathscr{G}_X(N)} \dot{V}(B)$, hence

$$\begin{aligned} m\big(\dot{T}(N) \cap \dot{X}\big) &\geq m\Big(\bigcup_{B \in \mathscr{G}_X(N)} \dot{V}(B)\Big) = \#\mathscr{G}_X(N) m\big(\dot{V}(B)\big) \\ &\sim \#\mathscr{G}_X(N)\psi(N)^{\delta} \gg m(\dot{X})\Big(\frac{\psi(N)}{\lambda(N)}\Big)^{\delta}, \end{aligned}$$

and the required result follows from (7).

Lemma 4. *There exist a constant K_2 and an integer N^* such that, for any $N \geq N^*$ and for any ball C with centre in L,*

$$m\big(\dot{T}(N) \cap \mathrm{cl}(C)\big) \leq K_2\big[m\big(\dot{T}(N)\big) m\big(\mathrm{cl}(5\,C)\big) + \psi(N)^{\delta}\big]. \tag{9}$$

Proof. Without loss of generality assume that C is closed, so $C = \mathrm{cl}(C)$. Let C_N^+ be the $4\lambda(N)$ neighbourhood of C, i.e. C_N^+ is a ball of radius $r(C_N^+) = r(C) + 4\lambda(N)$ concentric with C. Assume that $C \cap \dot{T}(N) \neq \emptyset$, thus $C \cap U \neq \emptyset$. Now let

$$\mathscr{G}_+(N) = \{B \in \mathscr{G}(N) : B \subset C_N^+\}.$$

Then if $B \cap C \neq \emptyset$ for B in $\mathscr{G}(N)$ then $B \in \mathscr{G}_+(N)$. By construction, for N sufficiently large to ensure that $5\lambda(N) < r_0$,

$$m(C_N^+) \geq m\Big(\bigcup_{B \in \mathscr{G}_+(N)} B\Big) \sim \#\mathscr{G}_+(N)\lambda(N)^{\delta}. \tag{10}$$

Now suppose, $\lambda(N) \le r(C)$, then $m(C_N^+) \le m(5C)$ since $r(C_N^+) \le 5r(C)$. It follows from (10) that $\#\mathscr{G}_+(N) \ll m(5C)\lambda(N)^{-\delta}$. On the other hand if $\lambda(N) > r(C)$, then $m(C_N^+) \ll \lambda(N)^\delta$ since $r(C_N^+) < 5\lambda(N) < r_0$. It follows from (10) that $\#\mathscr{G}_+(N) \ll 1$. In both cases, $m\big(\dot V(B) \cap C\big) \le m\big(\dot V(B)\big) \sim \psi(N)^\delta$. By these estimates and (7),

$$\begin{aligned}
m\big(\dot T(N) \cap C\big) &= m\Big(\bigcup_{B \in \mathscr{G}(N)} \dot V(B) \cap C\Big) \le m\Big(\bigcup_{B \in \mathscr{G}_+(N)} \dot V(B)\Big) \\
&\ll \frac{m(5C)}{\lambda(N)^\delta} m\big(\dot V(B)\big) + \psi(N)^\delta \\
&\ll m(5C)\Big(\frac{\psi(N)}{\lambda(N)}\Big)^\delta + \psi(N)^\delta \ll m(5C) m\big(\dot T(N)\big) + \psi(N)^\delta,
\end{aligned}$$

as required.

Since $\psi(N) \le \lambda(N) < 1$ for N sufficiently large, it follows that the ratio $\log \lambda(N)/\log \psi(N) \le 1$ and so

$$\gamma = \limsup \Big\{ \frac{\log \lambda(N)}{\log \psi(N)} : N = 1, 2, \dots \Big\} \le 1.$$

Suppose first there exists a strictly increasing sequence $\{N_r : r = 0, 1, 2, \dots\}$ such that

$$\lim_{r \to \infty} \frac{\lambda(N_r)}{\psi(N_r)^\gamma} = 0.$$

Since $\delta > 0$, when $r \to \infty$,

$$h(N_r) = \left(\frac{\lambda(N_r)}{\psi(N_r)^\gamma} \right)^\delta \to 0.$$

Let $\mathscr{C}$ be a collection of open balls, C with centres in L such that

$$\sum_{C \in \mathscr{C}} r(C)^{\delta\gamma} < 1. \tag{11}$$

It will be shown that there exists a positive number σ such that if $r(C) < \sigma$ for all balls C in $\mathscr{C}$, then $\mathscr{C}$ cannot cover the set T^∞. By the definition of the L-centred Hausdorff measure, it follows that $\mathscr{H}_L^{\delta\gamma}(T^\infty) \ge 1$ when the sequence $\{N_r\}$ and the limit above exist. The constant 1 in (11) can be replaced by an arbitrarily large number, so that in the case where the above limit condition holds, $\mathscr{H}_L^{\delta\gamma}(T^\infty) = \infty$. Hence $\dim_L T^\infty \ge \delta\gamma$ and so $\dim_L \Lambda(\psi) \ge \delta\gamma$.

Remark 2. Since $\mathscr{H}^s_L(T^\infty) \le \mathscr{H}^s_L\big(\Lambda(\psi)\big)$,

$$\mathscr{H}^{\delta\gamma}_L(T^\infty) = \infty \quad \Rightarrow \quad \mathscr{H}^{\delta\gamma}_L\big(\Lambda(\psi)\big) = \infty.$$

By Lemma 1, $\mathscr{H}^{\delta\gamma}_L(F) \sim \mathscr{H}^{\delta\gamma}(F)$ for any subset F of L. Hence, if there exists a strictly increasing sequence $\{N_r : r = 0,1,2,\ldots\}$ such that $\lambda(N_r)/\psi(N_r)^\gamma = 0$ as $r \to \infty$, then $\mathscr{H}^{\delta\gamma}\big(\Lambda(\psi)\big) = \infty$.

Let $\{r(s) : s = 0,1,2,\ldots\}$ be a strictly increasing infinite subsequence of integers with $r(C) \le \psi\big(N_{r(0)}\big)$ for each C in $\mathscr{C}$, and for each $s \ge 1$ let

$$\mathscr{C}_s = \big\{C \in \mathscr{C} : \psi\big(N_{r(s)}\big) < r(C) \le \psi\big(N_{r(s-1)}\big)\big\} \subset \mathscr{C},$$
$$E_s = \bigcup_{C \in \mathscr{C}_s} C.$$

For the remainder of the proof $r(0)$ will be taken to be sufficiently large so that $\psi\big(N_{r(0)}\big) \le r_0/10$. Hence (1) holds for all balls $10C$ of radius $10r(C)$ concentric with C in $\mathscr{C}$.

The sets E_s are open since the balls C in $\mathscr{C}$ are open. Write ψ_s for $\psi\big(N_{r(s)}\big)$, T_s for $T\big(N_{r(s)}\big)$ and h_s for $h\big(N_{r(s)}\big)$. Note that for all $s = 0,1,2,\ldots$, T_s is a closed subset of Ω and for any C in $\mathscr{C}$, $r(C) < \psi_0$.

Lemma 5. *If $r(0)$ is chosen sufficiently large so that (7) and (9) hold for all $N \ge N_{r(0)}$, then for each $s = 0,1,2,\ldots$*

$$m\big(\dot{T}_{s+1} \cap \mathrm{cl}(E_{s+1})\big) \le K_3 m(\dot{T}_{s+1})\big[m(\dot{T}_s)h_s + h_{s+1}\big], \tag{12}$$

where K_3 does not depend on $\mathscr{C}$ or on the sequence $\{N_{r(s)} : s = 0,1,2,\ldots\}$.

Proof. Since $N_{r(s)} \ge N_{r(0)}$ for $s \ge 0$, it follows from (9) that

$$\begin{aligned} m\big(\dot{T}_{s+1} \cap \mathrm{cl}(E_{s+1})\big) &\le \sum_{C \in \mathscr{C}_{s+1}} m\big(\dot{T}_{s+1} \cap \mathrm{cl}(2C)\big) \\ &\ll \sum_{C \in \mathscr{C}_{s+1}} \Big[m(\dot{T}_{s+1})\big(10r(C)\big)^\delta + (\psi_{s+1})^\delta\Big] \\ &\ll m(\dot{T}_{s+1}) \sum_{C \in \mathscr{C}_{s+1}} r(C)^\delta + \sum_{C \in \mathscr{C}_{s+1}} \psi^\delta_{s+1}. \end{aligned}$$

However from (7) and (11) it follows that

$$\sum_{C \in \mathscr{C}_{s+1}} r(C)^\delta \ll m(\dot{T}_s)h_s \qquad \text{and} \qquad \sum_{C \in \mathscr{C}_{s+1}} \psi^\delta_{s+1} \ll m(\dot{T}_{s+1})h_{s+1}.$$

Hence

$$m\big(\dot{T}_{s+1} \cap \mathrm{cl}(E_{s+1})\big) \ll m(\dot{T}_{s+1})\big[m(\dot{T}_s)h_s + h_{s+1}\big]$$

as required.

Now let $G_0 = T_0$ and define the set G_s $(\subseteq T_s)$ recursively as

$$G_s = (G_{s-1} \cap T_s)\backslash E_s, \qquad s = 1, 2, \ldots;$$

for each $s = 0, 1, 2, \ldots$, the set G_s is compact and contains G_{s+1}. Using induction, a positive lower bound for $m(G_s)$ will be obtained. Without loss of generality, take $K_1 \leq 1$ in (8).

Lemma 6. *The sequence of integers $r(s)$ can be chosen so that, for each $s = 0, 1, 2, \ldots$,*

$$m(G_s) \geq m(\dot{G}_s) \geq (\tfrac{1}{2}K_1)^{s+1} \prod_{j=0}^{s} m(\dot{T}_j) > 0. \tag{13}$$

Proof. Clearly $m(G_s) \geq m(\dot{G}_s)$ for all s, and (13) holds when $s = 0$. Now choose $r(0)$ sufficiently large so that (12) holds and $h_0 < (2K_3)^{-1}(K_1/2)^2$ (this is possible since $h(N_r) \to 0$ as $r \to \infty$). By Lemma 3, with $X = \dot{T}_0$, choose $r(1)$ large enough so that $N_{r(1)} > N_{r(0)}$, $h_1 < (2K_3)^{-1}(K_1/2)^3 m(\dot{T}_0)$ and $m(\dot{T}_0 \cap \dot{T}_1) \geq K_1 m(\dot{T}_0)m(\dot{T}_1)$. Then,

$$\begin{aligned}
m(\dot{G}_1) &\geq m(\dot{T}_0 \cap \dot{T}_1) - m\big(\dot{T}_0 \cap \dot{T}_1 \cap \mathrm{cl}(E_1)\big)\\
&\geq K_1 m(\dot{T}_0)m(\dot{T}_1) - K_3 m(\dot{T}_1)\big[m(\dot{T}_0)h_0 + h_1\big]\\
&\geq K_1 m(\dot{T}_0)m(\dot{T}_1) - \tfrac{1}{2}(\tfrac{1}{2}K_1)^2 m(\dot{T}_0)m(\dot{T}_1)[1 + \tfrac{1}{2}K_1]\\
&\geq (\tfrac{1}{2}K_1)^2, m(\dot{T}_0)m(\dot{T}_1) > 0,
\end{aligned}$$

and hence (13) holds when $s = 1$.

Now suppose that for $n \geq 2$ a strictly increasing sequence of integers $r(s)$, $s = 1, 2, \ldots, n$, has been chosen such that (13) and

$$h_s \leq (2K_3)^{-1}(\tfrac{1}{2}K_1)^{s+2} \prod_{j=0}^{s-1} m(\dot{T}_j) \tag{14}$$

hold. Again using Lemma 3, with $X = G_n$, it is possible to choose $r(n+1) > r(n)$ sufficiently large so that (14) holds for $s = n + 1$, and

$$m(\dot{G}_n \cap \dot{T}_{n+1}) \geq K_1 m(\dot{G}_n)m(\dot{T}_{n+1}).$$

Then by (13), (14) and Lemma 5

$$\begin{aligned}
m(\dot{G}_{n+1}) &\geq m(\dot{G}_n \cap \dot{T}_{n+1}) - m\big(\dot{G}_n \cap \dot{T}_{n+1} \cap \mathrm{cl}(E_{n+1})\big)\\
&\geq K_1 m(\dot{G}_n)m(\dot{T}_{n+1}) - K_3 m(\dot{T}_{n+1})\big[m(\dot{T}_n)h_n + h_{n+1}\big]\\
&\geq K_1(\tfrac{1}{2}K_1)^{n+1} \prod_{j=0}^{n+1} m(\dot{T}_j) - \tfrac{1}{2}(\tfrac{1}{2}K_1)^{n+2} \prod_{j=0}^{n+1} m(\dot{T}_j)\big[1 + \tfrac{1}{2}K_1\big]\\
&\geq (\tfrac{1}{2}K_1)^{n+2} \prod_{j=0}^{n+1} m(\dot{T}_j) > 0,
\end{aligned}$$

and so (13) holds for $s = n+1$. The result now follows by induction.

Thus for each $s = 0, 1, 2, \ldots$, the compact set G_s is non-empty and so the set $G_\infty = \bigcap_{s=0}^{\infty} G_s$ is non-empty. Moreover $r(C) < \psi_0$ for each C in $\mathscr{C}$, so by construction $\mathscr{C} = \bigcup_{s=1}^{\infty} \mathscr{C}_s$. Also if $\mathbf{x} \in C$ for some C in $\mathscr{C}$, then $C \in \mathscr{C}_s$ for some s and $\mathbf{x} \in E_s$, whence $\mathbf{x} \notin G_s \supset G_\infty$. Thus the collection $\mathscr{C}$ does not cover the set G_∞. Since

$$G_\infty \subset \bigcap_{s=1}^{\infty} T_s \subset T^\infty$$

this proves Theorem 1 when there exists a strictly increasing sequence $\{N_r : r = 0, 1, 2, \ldots\}$ such that $\lambda(N_r)/\psi(N_r)^\gamma \to 0$ as $r \to \infty$. Now suppose that no such sequence $\{N_r\}$ can be chosen. Then $\gamma > 0$ since $\lambda(N) \to 0$ as $N \to \infty$. Choose any η in $(0, \gamma)$ and let $\widehat{\gamma} = \gamma - \eta > 0$. Then by the definition of lim sup, there exists a sequence $\{N_r\}$ such that

$$\lim_{r\to\infty} \frac{\lambda(N_r)}{\psi(N_r)^{\widehat{\gamma}}} = 0.$$

Since $\delta > 0$, when $r \to \infty$,

$$\hat{h}(N_r) = \Big(\frac{\lambda(N_r)}{\psi(N_r)^{\widehat{\gamma}}}\Big)^\delta \to 0.$$

By repeating the above proof with h and γ replaced by $\hat{h}$ and $\widehat{\gamma}$ respectively, it can be shown that

$$\dim_L \Lambda(\psi) \geq \delta\widehat{\gamma} = \delta(\gamma - \eta).$$

Since η can be made arbitrarily small, it follows that $\dim_L \Lambda(\psi) \geq \delta\gamma$, and the proof of Theorem 1 is complete.

The following theorem is a more general result for families of m-ubiquitous systems and is an analogue of Theorem 2 in [4].

Theorem 2. *For each $i = 1, 2, \ldots$, suppose that $(\mathscr{R}^i, \beta^i)$ is an m-ubiquitous system with respect to λ^i and that $\psi^i \colon \mathbf{R}^+ \to \mathbf{R}^+$ is a decreasing function. Then*

$$\dim \bigcap_{i=1}^{\infty} \Lambda(\psi^i) \geq \delta \inf\{\gamma^i : i = 1, 2, \ldots\} = \delta\gamma^\infty$$

where

$$\gamma^i = \min\left\{1, \limsup_{N\to\infty} \frac{\log \lambda^i(N)}{\log \psi^i(N)}\right\}.$$

If there exist strictly increasing sequences $\{N_r^i : r = 0, 1, 2, \ldots\}$ such that

$$\lambda^i(N_r^i)/\psi^i(N_r^i)^{\gamma^i} \to 0$$

as $r \to \infty$ *for all those* i *such that* $\gamma^i = \gamma^\infty$, *then*

$$\mathscr{H}^{\delta\gamma^\infty}\Big(\bigcap_{i=1}^{\infty}\Lambda(\psi^i)\Big) = \infty.$$

Proof. In view of Theorem 1, the proof follows the same line of argument as the proof of Theorem 2 in [4].

4. Applications

The Euclidean norm of vectors $\mathbf{x} = (x_1, \ldots, x_k, x_{k+1})$ in $\mathbf{R}^{k+1}$ will be denoted by $\|\mathbf{x}\|$. The unit ball $B^{k+1} = \{\mathbf{x} \in \mathbf{R}^{k+1} : \|\mathbf{x}\| < 1\}$, is a model of $k+1$-dimensional hyperbolic space and supports a metric s derived from the differential

$$ds = \frac{\|d\mathbf{x}\|}{1 - \|\mathbf{x}\|^2}.$$

An alternative model for $k+1$-dimensional hyperbolic space is the upper half space model $\mathbf{H}^{k+1} = \{\mathbf{x} \in \mathbf{R}^{k+1} : x_{k+1} > 0\}$, which supports a metric ϱ derived from the differential

$$d\varrho = \frac{\|d\mathbf{x}\|}{2|x_{k+1}|}.$$

For $\mathbf{z}, \mathbf{w}$ in B^{k+1} let

$$L(\mathbf{z}, \mathbf{w}) = \tfrac{1}{2} + \frac{\|\mathbf{z} - \mathbf{w}\|^2}{\big(1 - \|\mathbf{z}\|^2\big)\big(1 - \|\mathbf{w}\|^2\big)},$$

and for $\mathbf{z}, \mathbf{w}$ in $\mathbf{H}^{k+1}$ let

$$L_{\mathbf{H}}(\mathbf{z}, \mathbf{w}) = \tfrac{1}{2} + \frac{\|\mathbf{z} - \mathbf{w}\|^2}{4|z_{k+1}||w_{k+1}|}.$$

Let G be a convex co-compact group preserving the unit ball and let $L(G)$ denote the limit set of G, the set of cluster points in the unit sphere S^k of any orbit of G in B^{k+1}. By definition (see [9], [12]), the action of G on the convex hull of $L(G)$ has a compact fundamental region in B^{k+1}. For $s \in \mathbf{R}$ define the exponent of convergence of G as

$$\delta(G) = \inf\Big\{ s > 0 : \sum_{g \in G} L\big(\mathbf{z}, g(\mathbf{w})\big)^{-s} < \infty \Big\}.$$

If G is of the first kind, then $L(G) = S^k$ and it follows that $\dim L(G) = k$. However, if G is of the second kind the limit set is of zero Lebesgue measure and $\dim L(G) = \delta(G)$. In fact these statements are true for any arbitrary non-elementary geometrically finite group [14]. Associated with $L(G)$ is the Patterson measure μ [10], which is a non-atomic, ergodic probability measure supported on $L(G)$. Furthermore, for convex co-compact groups, any small Euclidean ball $B(c, r)$ in S^k, with centre c in $L(G)$ has μ measure comparable to $r^{\delta(G)}$.

Lemma 7. *Let G be a convex co-compact group. Then there are constants a, b, r_0 dependent only on G, such that if $c \in L(G)$ and $r \le r_0$ then*

$$ar^{\delta(G)} < \mu\big(B(c,r)\big) < br^{\delta(G)}.$$

This is proved in [9], [12] and [14]. For a full and detailed account of the concepts and results mentioned above, the reader is referred to [9] and [12]. For convex co-compact groups, Lemma 7 implies that the Patterson measure satisfies (1) with $L = L(G)$ and $\delta = \delta(G)$.

4.1. Orbital approximation. For each real number τ and an arbitrary hyperbolic fixed point $\mathbf{y}$ ($\in S^k$) of G, let $W_{G;\mathbf{y}}(\tau)$ denote the set of points $\mathbf{x}$ in $L(G)$ which are τ-approximable with respect to $\mathbf{y}$ (Definition 1, [15], [16]), i.e.,

$$W_{G;\mathbf{y}}(\tau) = \big\{\mathbf{x} \in L(G) : \|\mathbf{x} - g(\mathbf{y})\| \le L(\mathbf{0}, g(\mathbf{0}))^{-\tau} \text{ for infinitely many } g \text{ in } G\big\}.$$

It is easy to verify that $L(\mathbf{0}, g(\mathbf{0}))$ is comparable with $1/\big(1 - \|g(\mathbf{0})\|\big)$ which in turn is comparable with the exponential of the hyperbolic distance between the orbit point $g(\mathbf{0})$ and the origin.

By using the properties of the Patterson measure μ and μ-ubiquity, the hyperbolic analogue of the Jarník–Besicovitch theorem is extended from non-elementary geometrically finite groups of the first kind (Theorem 2, [8] and [16]) to convex co-compact groups. Clearly this is only an extension for groups without parabolic elements.

Theorem 3. *Let G be a convex co-compact group and let $\mathbf{y}$ be an arbitrary hyperbolic fixed point of G. For $\tau > 1$*

$$\dim W_{G;\mathbf{y}}(\tau) = \frac{\delta(G)}{\tau}.$$

Proof. By Theorem 1 in [16], it is sufficient to show that $\dim W_{G;\mathbf{y}}(\tau) \ge \delta(G)/\tau$ ($\tau > 1$). First some notation is needed. Let $C = \mathbf{H}^{k+1} \cap S^k$, and without loss of generality assume that $\mu(C) \neq 0$. In order to simplify notation, write $L_g = L\big(\mathbf{0}, g(\mathbf{0})\big)$.

The lower bound for $\dim W_{G;\mathbf{y}}(\tau)$ will now be obtained by using the notion of m-ubiquity with $m = \mu$,

$$\Omega = C \cap L(G), \quad J = G, \quad R_\alpha = g(\mathbf{y}), \quad \beta_\alpha = \frac{L_g}{k_1}, \quad \psi(x) = (k_1 x)^{-\tau}$$

and then applying Theorem 1 with $\delta = \delta(G)$. It follows that $\Lambda(\psi)$ is a subset of $W_{G;\mathbf{y}}(\tau)$ and a lower bound for $\dim \Lambda(\psi)$ is also a lower bound for $\dim W_{G;\mathbf{y}}(\tau)$.

Let $P = \{\mathbf{y}, \mathbf{z}\}$ be a pair of fixed points of a hyperbolic element of G. Denote by $G_{\mathbf{yz}}$ the stabilizer of $\mathbf{y}$ or equivalently of $\mathbf{z}$ (the stabilizer of $\mathbf{y}$ is equal to the stabilizer of $\mathbf{z}$ since G is discrete), and let $G\|G_{\mathbf{yz}}$ be a set of representatives of the cosets $\{gG_{\mathbf{yz}} : g \in G\}$ so chosen that if $g \in G\|G_{\mathbf{yz}}$, $h \in G_{\mathbf{yz}}$ then $L_g \le L_{gh}$. In this notation the minimum of L_{gh} ($h \in G_{\mathbf{yz}}$) occurs when $gh \in G\|G_{\mathbf{yz}}$.

In view of Theorem 4 in [16] and the definition of $G\|G_{\mathbf{yz}}$, there is a positive constant k_2 with the following property: for each $\mathbf{x}$ in Ω, $N > 1$, there exist $\mathbf{w} = \mathbf{w}_{(\mathbf{x},N)}$ in P, $g = g_{(\mathbf{x},N)}$ in $G\|G_{\mathbf{yz}}$ with $L_g < N$ so that

$$\|\mathbf{x} - g(\mathbf{w})\| < \frac{k_2}{N}. \tag{15}$$

Let

$$E(N) = \left\{\mathbf{x} \in \Omega : L_{g(\mathbf{x},N)} < \frac{N}{\log N}\right\}$$

and

$$A(N) = \{\mathbf{x} \in \Omega : \mathrm{dist}_\infty(\mathbf{x}, \partial\Omega) \gg N^{-1}\log N\} \setminus E(N),$$

so that for each $\mathbf{x}$ in $A(N)$, the point $g(\mathbf{w})$ satisfying (15) lies in Ω. Here $\partial\Omega$ is the boundary of Ω with respect to L. It is easy to see that

$$E(N) \subset \bigcup_{\mathbf{w}\in P} \bigcup_{\substack{g\in G\\ L_g < N/\log N}} B\big(g(\mathbf{w}), k_2 N^{-1}\big)$$

and by Lemma 7, for sufficiently large N,

$$\mu\big(B\big(g(\mathbf{w}), k_2 N^{-1}\big)\big) \sim N^{-\delta(G)}.$$

By Theorem 3 in [15],

$$\mu\big(E(N)\big) \ll \sum_{\substack{g\in G\\ L_g < N/\log N}} \Big(\frac{1}{N}\Big)^{\delta(G)} \ll \frac{1}{N^{\delta(G)}}\Big(\frac{N}{\log N}\Big)^{\delta(G)}.$$

Hence for N large, $\mu\big(E(N)\big) \ll (\log N)^{-\delta(G)}$, so that $\mu\big(\Omega\backslash A(N)\big) \to 0$ as $N \to \infty$. It follows from the definition of $A(N)$ that for each $\mathbf{x}$ in $A(N)$, there exist $\mathbf{w}_{(\mathbf{x},N)}$ in P, $g_{(\mathbf{x},N)}$ in $G\|G_{\mathbf{yz}}$ such that (15) is satisfied and

$$\frac{N}{\log N} \le L_g \le N. \tag{16}$$

Suppose first that $\mathbf{w} = \mathbf{z}$, then by (16) and Proposition 2 in [16], there exists an element h in $G\|G_{\mathbf{yz}}$ such that

$$\|g(\mathbf{z}) - h(\mathbf{y})\| \le k_3\frac{\log N}{N} \qquad \text{and} \qquad k_4\frac{N}{\log N} \le L_h \le k_1 N.$$

On using the triangle inequality and (15), it is easily verified that

$$\|\mathbf{x} - h(\mathbf{y})\| \le k_5 N^{-1} \log N$$

where $k_5 = k_2 + k_3$ and $L_h \le k_1 N$. Thus for each $\mathbf{x}$ in $A(N)$ there exists an

$$\alpha = g_{(\mathbf{x},N)} \in J(N) = \{g \in G : L_g \le k_1 N\}$$

such that

$$\|\mathbf{x} - R_\alpha\| \le k_5 N^{-1} \log N, \tag{17}$$

where $R_\alpha = g(\mathbf{y})$ lies in Ω. From now on write g instead of h. In the case of $\mathbf{w} = \mathbf{y}$, it is easy to see that (17) is satisfied for some g in $J(N)$. Thus

$$A(N) \subset \bigcup_{\substack{g \in G \| G_{\mathbf{yz}} \\ L_g < k_1 N}} B\big(g(\mathbf{y}), k_5 N^{-1} \log N\big) \subset \bigcup_{g \in J(N)} \tfrac{1}{3} B\big(g(\mathbf{y}), \lambda(N)\big),$$

where $\lambda(N) = 3k_5 N^{-1} \log N$, whence

$$\mu\Big(\Omega \setminus \bigcup_{g \in J(N)} \tfrac{1}{3} B\big(g(\mathbf{y}), \lambda(N)\big)\Big) \to 0$$

as $N \to \infty$. This verifies that the system $(\mathscr{R}, \beta)$ is μ-ubiquitous relative to the function λ. The proof can now be completed by applying Theorem 1, since $\gamma = 1/\tau$ and $\delta = \delta(G)$.

Remark 3. Let A be a non-empty set of hyperbolic fixed points of G. In view of Theorem 3 and the fact that

$$\dim\Big(\bigcup_{\mathbf{y} \in A} W_{G;\mathbf{y}}(\tau)\Big) = \max_{\mathbf{y} \in A}\{\dim W_{G;\mathbf{y}}(\tau)\}$$

for any finite set A, it is easy to see that when $\tau > 1$, the set of τ-approximable points with respect to A (Definition 1, [15], [16]) has Hausdorff dimension $\delta(G)/\tau$, i.e.

$$\dim\Big(\bigcup_{\mathbf{y} \in A} W_{G;\mathbf{y}}(\tau)\Big) = \frac{\delta(G)}{\tau}.$$

In view of Theorem 2, which deals with the intersection of families of m-ubiquitious systems, it is easy to verify that for $\tau > 1$,

$$\dim\Big(\bigcap_{\mathbf{y} \in A} W_{G;\mathbf{y}}(\tau)\Big) = \frac{\delta(G)}{\tau}.$$

4.2. Geometric results. In this subsection, let G be a convex co-compact group acting in the upper half plane $\mathbf{H}^2$ model. The unit disk B^2 and the upper half plane $\mathbf{H}^2$ models are 'equivalent' and Theorem 3 clearly translates to the latter with $L_g = L_{\mathbf{H}}(\mathbf{e}_2, g(\mathbf{e}_2))$, where $\mathbf{e}_2 = (0,1)$.

Let $\mathscr{M} = \mathbf{H}^2/G$ denote the associated complete, non-compact Riemann surface of constant negative sectional curvature. It is well known that

$$\mathscr{M} = X_o \cup \bigcup_{i=1}^{n} Z_i,$$

where X_o is compact and each Z_i is isometric to $S^1 \times [0, +\infty)$ with respect to the metric $dr^2 + \cosh^2 r \, d\theta^2$ [10]. The Z_i's are usually referred to as funnels. The infimum of the lengths of the curves in the non-trivial free homotopy classes on each funnel is positive and equals the length of the simple closed geodesic $\mathscr{G}$ limiting the funnel.

Given a point p on $\mathscr{M}$, let $S(p)$ be the unit disc in the tangent space of $\mathscr{M}$ centred at p, and for every direction v in $S(p)$ let γ_v be the geodesic emanating from p in the direction v. Finally, for t in $\mathbf{R}$, let $\gamma_v(t)$ denote the point achieved after traveling time t along γ_v. With this notation in mind, the geometric result can now be stated as follows:

Theorem 4. *Let* dist *be the distance in* $\mathscr{M}$ *and let* $\mathscr{G}$ *denote a simple closed geodesic.*

(1) *For* $0 \le \alpha \le 1$, *if* $\mathscr{G}$ *is limiting a funnel*

$$\dim\Big\{v \in S(p) : \limsup_{t\to\infty} \frac{-\log\big(\operatorname{dist}(\gamma_v(t), \mathscr{G})\big)}{t} \ge \alpha\Big\} = \frac{1-\alpha}{1+\alpha}\delta.$$

(2) *With respect to* μ, *for almost all directions* v *in* $S(p)$

$$\limsup_{t\to\infty} \frac{-\log\big(\operatorname{dist}(\gamma_v(t), \mathscr{G})\big)}{\log t} = \frac{1}{2\delta}.$$

Remark 4. As will become apparent in the proof, Part 1 of the above theorem is the geometric interpretation of Theorem 3 with $k = 1$. For $\alpha = 0$, the statement of Part 1 is trivial, since the set under consideration is the whole limit set. Part 2 is a 'Khinchin-like' result, analogous to Theorem 6 [13], and is a refinement of Part 1 in the case of $\alpha = 0$.

Proof of Theorem 4. Part 1. If γ_v satisfies the inequality

$$\limsup_{t\to\infty} \frac{-\log\big(\operatorname{dist}(\gamma_v(t), \mathscr{G})\big)}{t} \ge \alpha, \tag{18}$$

then there exists a sequence $\{t_i : i \in \mathbf{N}\}$ tending to infinity such that

$$\operatorname{dist}\bigl(\gamma_v(t_i), \mathscr{G}\bigr) < e^{-\alpha t_i} \tag{19}$$

Let $\widetilde{\gamma}_v$ be a lift of γ_v into $\mathbf{H}^2$ with endpoint ξ in $\mathbf{R}$, and let $\widetilde{G}_i$ be the lift of $\mathscr{G}$ such that $\widetilde{G}_i$ is closer to the point $\widetilde{\gamma}_v(t_i)$ than any other lift of $\mathscr{G}$. Finally, let $\{\eta_i^-, \eta_i^+\}$ be the endpoints of $\widetilde{G}_i$ with η_i^- closer to ξ than η_i^+.

Since $\mathscr{G}$ is limiting a funnel, ξ, η_i^-, η_i^+ are co-linear. Then for t_i sufficiently large

$$\frac{r_i}{R_i} \sim d_i^2 \qquad \text{and} \qquad r_i R_i \sim e^{-2t_i}, \tag{20}$$

where $d_i = \varrho\bigl(\widetilde{\gamma}_v(t_i), \widetilde{G}_i\bigr)$, $R_i = \frac{1}{2}\|\eta_i^- - \eta_i^+\|$ and $r_i = \|\xi - \eta_i^-\|$. It follows from (20) that inequality (19) is equivalent to

$$r_i \leq k_6 R_i^{(1+\alpha)/(1-\alpha)} \tag{21}$$

where $k_6 = k_6(p)$ is a positive constant. Hence, if ξ is not an endpoint of any lift of $\mathscr{G}$, there are infinitely many solutions of (21). The argument above is clearly reversible, hence if (21) has infinitely many solutions then the projection of the geodesic $\widetilde{\gamma}_v$ to $\mathscr{M}$ satisfies (18). A simple calculation shows that if $\{g(\eta^-), g(\eta^+)\}$ are the end points of a lift of $\mathscr{G}$ with $g \in G \| G_{\eta^- \eta^+}$, then

$$L_g = L_{\mathbf{H}}\bigl(\mathbf{e}_2, g(\mathbf{e}_2)\bigr) \sim \|g(\eta^-) - g(\eta^+)\|^{-1}.$$

Hence, part one of Theorem 4 follows as a consequence of Theorem 3 with $\tau = (1+\alpha)/(1-\alpha)$.

Part 2. Let $\widetilde{\mathscr{G}}$ be a lift of $\mathscr{G}$ with end points $\{\eta^-, \eta^+\}$ and for g_i in G, let $\{\eta_i^-, \eta_i^+\}$ and R_i be the endpoints and the euclidean radius of $g_i(\widetilde{\mathscr{G}})$ respectively. For an arbitrary positive ε, let $C_\varepsilon(\mathscr{G})$ be the set of points in $\mathscr{M}$ within a distance ε of $\mathscr{G}$, and let $C_\varepsilon(\widetilde{\mathscr{G}})$ be the corresponding ε-neighbourhood of $\widetilde{\mathscr{G}}$, i.e.

$$C_\varepsilon(\widetilde{\mathscr{G}}) := \bigl\{x \in \mathbf{H}^2 : \varrho(x, \widetilde{\mathscr{G}}) < \varepsilon\bigr\}.$$

For obvious geometric reasons, the set $C_\varepsilon(\mathscr{G})$ is usually referred to as a collar. It is a well known fact that for $\varepsilon = \varepsilon(\mathscr{G})$ sufficiently small, the countable union

$$\bigcup_{g_i \in G \setminus G_{\eta^- \eta^+}} g_i\bigl(C_\varepsilon(\widetilde{\mathscr{G}})\bigr)$$

is pairwise disjoint.

On combining various ideas contained in the proof of Theorems 4.3.2, 4.5.1 and 4.6.5 in [9], and using similar arguments to those in the proof of Lemmas 1.2 and 1.3 in [8], the following can be deduced. Given a closed interval $\mathscr{B}$ in $\mathbf{R}$ with centre in $L(G)$ there exists a number ϱ in $(0,1)$ with the following property: the number ν_n of ε-neighbourhoods $g_i\big(C_\varepsilon(\widetilde{\mathscr{G}})\big)$ in $\mathbf{H}^2$ with at least one of its end points in $\mathscr{B}$ and radii $R_i \in [\varrho^{n+1}, \varrho^n)$, satisfies for all $n \geq n_0(G, \mathscr{B}, \mathscr{G})$

$$\nu_n \sim \left(\frac{1}{\varrho^n}\right)^\delta \mu(\mathscr{B}), \tag{22}$$

where the implied constants depend only on G and $\mathscr{G}$. The above comparability is central to the proof.

It is easy to verify that there exists a positive $\chi = \chi(\varepsilon, \varrho) < \cosh\varepsilon - 1$, such that for all R_i, $R_j \in [\varrho^{n+1}, \varrho^n)$ with $i \neq j$

$$B(c_i, \chi R_i) \cap B(c_j, \chi R_j) = \emptyset, \tag{23}$$

where $c_i \in \{\eta_i^-, \eta_i^+\}$ and $c_j \in \{\eta_j^-, \eta_j^+\}$.

Let $a : [1, \infty) \longrightarrow (0, \chi]$ be a non-increasing continuous function with the property that $a(x_1) \sim a(x_2)$, for $1/x_1, 1/x_2 \in [\varrho^{n+1}, \varrho^n)$. Now, for a positive integer K, consider the following sets:

$$A_{n,K}(a) = \bigcup_{\substack{g_i \in G \\ R_i \in [\varrho^{n+1}, \varrho^n)}} B\big(c_i, Ka(R_i^{-1})R_i\big)$$
$$A_{\infty,K}(a) = \limsup_{n\to\infty} A_{n,K}(a)$$

and let

$$A_\infty(a) = \bigcup_{K=1}^{\infty} A_{\infty,K}(a).$$

It follows from (22) and (23), that the sets $\{A_{n,K}(a) : n = n_0, n_0+1, \ldots\}$ are pairwise quasi-independent; i.e. there is a positive constant k_7 so that for $n_0 < i < j$,

$$\mu\big(A_{i,K}(a) \cap A_{j,K}(a)\big) \leq k_7 \mu\big(A_{i,K}(a)\big)\mu\big(A_{j,K}(a)\big).$$

Hence on using (22), the Borel–Cantelli lemma and its pairwise quasi-independent version (Proposition 2 [13]) as in Section 3 and 4 of [13], it follows that

$$\int_1^\infty \frac{\big(a(x)\big)^\delta}{x}\,dx < \infty \implies \mu\big(A_{\infty,K}(a)\big) = 0, \quad \text{for all } K$$
$$\int_1^\infty \frac{\big(a(x)\big)^\delta}{x}\,dx = \infty \implies \mu\big(A_{\infty,K}(a)\big) > 0, \quad \text{for all } K.$$

Since $A_\infty(a)$ is G-invariant and the action of G on $L(G)$ is ergodic with respect to μ (see [9]), the set $A_\infty(a)$ has either zero or full μ-measure. Note that the results outlined in the above discussion constitute the convex co-compact analogue of Patterson's 'Khinchin-like' result for groups of the first kind, Theorem 9.1 [11]. The reader is referred to [17] for a Hausdorff dimensional 'interpretation' of the above discussion, in which the integral $\int_1^\infty \big(a(x)\big)^\delta/x\,dx$ is replaced by the lower order at infinity of the function $1/a$ and the Patterson measure of the set $A_\infty(a)$ is replaced by its Hausdorff dimension.

Now for any real positive $\sigma \le \delta$, define the function

$$a_\sigma(x) = \begin{cases} \dfrac{\chi}{(\log x)^{1/\sigma}} & \text{if } x \in [e, +\infty) \\ \chi & \text{if } x \in [1, e)\ . \end{cases}$$

It follows from (20) that

$$A_\infty(a_\delta) \subseteq \left\{ v : \limsup_{t\to\infty} \frac{-\log\big(\mathrm{dist}\big(\gamma_v(t), \mathscr{G}\big)\big)}{\log t} \ge \frac{1}{2\delta} \right\}, \tag{24}$$

and

$$\bigcap_{0<\sigma<\delta} \big(A_\infty(a_\sigma)\big)^c \subseteq \left\{ v : \limsup_{t\to\infty} \frac{-\log\big(\mathrm{dist}\big(\gamma_v(t), \mathscr{G}\big)\big)}{\log t} \le \frac{1}{2\delta} \right\}, \tag{25}$$

where $\big(A_\infty(a_\sigma)\big)^c$ denotes the complement of the set $A_\infty(a_\sigma)$.

From the above discussion $\mu\big(A_\infty(a_\delta)\big) = 1$, whence the set appearing on the right hand side of (24) has full μ-measure. For $\sigma < \delta$, the set $A_\infty(a_\sigma)$ has zero μ-measure, whence $\big(A_\infty(a_\sigma)\big)^c$ has full μ-measure. By construction,

$$\big(A_\infty(a_{\sigma_1})\big)^c \supset \big(A_\infty(a_{\sigma_2})\big)^c$$

for $\sigma_1 < \sigma_2$, hence the set appearing on the left hand side of (25) has full μ-measure. This implies that the set appearing on the right hand side of (25) also has full μ-measure. Thus the proof of Part 2 is now complete.

Remark 5. In (20) the inequalities

$$\frac{r_i}{R_i} \gg d_i^2 \qquad \text{and} \qquad r_i R_i \sim e^{-2t_i},$$

hold in higher dimensions for any simple closed geodesic $\mathscr{G}$ on $\mathscr{M}^{k+1} = \mathbf{H}^{k+1}/G$. Here G is a convex co-compact group acting in $\mathbf{H}^{k+1}$. It is these inequalities that are used in establishing (24) and (25). In view of this, the statement of Part 2 is valid in higher dimensions.

An analogous statement to Part 1 in higher dimensions can be established in the following way. Let $\mathscr{G}$ denote a simple directed closed geodesic of $\mathscr{M}^{k+1} = B^{k+1}/G$, where G is a convex co-compact group acting in the unit ball model B^{k+1}. For any point $\mathbf{x}$ on $\mathscr{G}$ consider the unit ball $S(\mathbf{x})$ in the tangent space of $\mathscr{M}^{k+1}$ centred at $\mathbf{x}$. Let $\zeta_{\mathscr{G}}(\mathbf{x})$ be the unit tangent vector to $\mathscr{G}$ at $\mathbf{x}$. With this in mind, let

$$\Omega_{\mathscr{G}} = \bigl\{\bigl(\mathbf{x}, \zeta_{\mathscr{G}}(\mathbf{x})\bigr) : \mathbf{x} \in \mathscr{G}\bigr\}.$$

Then for a given point $\mathbf{p}$ on $\mathscr{M}^{k+1}$ and α in $[0,1]$,

$$\dim\left\{ v \in S(\mathbf{p}) : \limsup_{t\to\infty} \frac{-\log\bigl(\operatorname{dist}\bigl(\bigl(\gamma_v(t), \gamma_v'(t)/\|\gamma_v'(t)\|\bigr), \Omega_{\mathscr{G}}\bigr)\bigr)}{t} \geq \alpha \right\} = \frac{1-\alpha}{1+\alpha}\delta,$$

where dist is the canonical invariant distance in the unit tangent space of $\mathscr{M}^{k+1}$ induced by the metric given, for example, in Theorem 8.1.1 [9].

Finally, let $\{\mathscr{G}_l\}_{l=1}^n$ be a collection of non-intersecting, simple closed geodesics on $\mathscr{M}$ limiting the funnels $\{Z_l\}_{l=1}^n$. Given $\mathscr{G}_l$, let T_l be the set of times t such that $\gamma_v(t)$ lies in $C_\varepsilon(\mathscr{G}_l)$. In view of Remark 3, the following generalization of Theorem 4 (Part 1) can be obtained.

Corollary 1. *For $0 \leq \alpha \leq 1$, the Hausdorff dimension of the set of directions v in $S(p)$ such that*

$$\limsup_{\substack{t\to\infty,\\ t\in T_l}} \frac{-\log\bigl(\operatorname{dist}\bigl(\gamma_v(t), \mathscr{G}_l\bigr)\bigr)}{t} \geq \alpha \qquad \textit{for all } l \in \mathscr{L} \subseteq \{1,\dots,n\},$$

is equal to $(1-\alpha)/(1+\alpha)\delta$.

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