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AN IDEAL BOUNDARY FOR DOMAINS IN n-SPACE

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Abstract. The Royden ideal boundary of a domain is the set of all points in the maximal ideal space of the domain's Royden algebra that do not lie in the domain. Elements of the Royden ideal boundary can be characterized as nets convergent in both the weak* and Euclidean topologies that have no subnet which is a sequence. As with other function algebras, boundary fibers can be defined as subsets of all points in the ideal boundary that project onto a single Euclidean boundary point. Quasiconformal mappings have homeomorphic extensions if and only if the adjoint of the corresponding Royden algebra isomorphism maps boundary fibers to boundary fibers. For domains with certain homogeneity properties, all boundary fibers are homeomorphic; and for domains finitely connected on the boundary, fibers are equivalent to prime ends. In any case, each boundary fiber has a complicated topology which contains the complement of the natural numbers in the Stone–Cech compactification of the natural numbers.

Introduction

Lewis [L] showed that two domains, Ω and Ω', in Euclidean n-space are quasiconformally equivalent if and only if the corresponding Royden algebras $A(\Omega)$ and $A(\Omega')$ are isomorphic. His proof extends ideas of Nakai [N] who proved the theorem for domains in two dimensions. It uses methods of functional analysis to characterize the maximal ideal space Ω^* of $A(\Omega)$ as a compact space containing Ω. For this reason, Ω^* is called the *Royden compactification* of Ω, and the set $\Omega^* \setminus \Omega$ is called the *Royden ideal boundary* of Ω. Lewis also showed that a quasiconformal mapping between two domains induces a homeomorphism between their Royden compactifications which preserves Royden ideal boundaries. Consequently, one might hope to find necessary conditions for the existence of such quasiconformal mappings by studying these boundaries.

We wish to examine the structure of the Royden ideal boundary from several different points of view and present some of its unusual properties. Borrowing ideas previously applied to the Banach algebra H^∞ [H], [Ga], we develop a theory of fibers in Ω^* over boundary points of Ω. We show that a quasiconformal mapping $f\colon \Omega \to \Omega'$ has a homeomorphic extension from $\partial\Omega$ to $\partial\Omega'$ if and only if the adjoint of the corresponding Royden algebra isomorphism maps boundary fibers in Ω^* onto boundary fibers in $\Omega^{*\prime}$. We discuss domains quasiconformally

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homogeneous with respect to the boundary and prove that all fibers in such domains are homeomorphic to each other; but in the special case that Ω is the unit ball, we prove that the Royden boundary is *not* the natural product of $\mathbf{S}^{n-1}$ with any particular fiber. We prove that if a domain is finitely connected on its boundary, then components of fibers correspond to the domain's prime ends. Finally, to exhibit the complicated topology of boundary fibers, we discuss $\beta\mathbf{N}$, the Stone–Cech compactification of the natural numbers, and show that each fiber contains a homeomorphic copy of $\beta\mathbf{N}\setminus\mathbf{N}$.

1. Preliminaries

$\mathbf{N}$ denotes the natural numbers; $\mathbf{R}$, the real numbers; $\mathbf{R}^n$, Euclidean n-space; and Ω and Ω', domains in $\mathbf{R}^n$ for $n\ge 2$. $\mathbf{B}(x,r)$ and $\mathbf{S}(x,r)$ denote the ball and sphere of radius r centered at x, while $\mathbf{B}^n$ and $\mathbf{S}^{n-1}$ denote the unit ball and sphere. In the context of integration, dx and dy designate n-dimensional Lebesgue measure. $\overline{\mathbf{R}}^n$ denotes $\mathbf{R}^n\cup\{\infty\}$, the one-point compactification of $\mathbf{R}^n$, and $\partial\Omega$ refers to the boundary of Ω in $\overline{\mathbf{R}}^n$. If $\{x_i\}$ is a sequence in Ω, we write $x_i\to\partial\Omega$ whenever $\{x_i\}$ has no cluster point in Ω.

The symbols $L^1_p(\Omega)$ and $L^1_{p,\mathrm{loc}}(\Omega)$ denote the spaces of functions whose weak partial derivatives exist and belong to $L^p(\Omega)$ and $L^p_{\mathrm{loc}}(\Omega)$ respectively. $W^1_p(\Omega)$ and $W^1_{p,\mathrm{loc}}(\Omega)$ denote the function spaces $L^p(\Omega)\cap L^1_p(\Omega)$ and $L^p_{\mathrm{loc}}(\Omega)\cap L^1_{p,\mathrm{loc}}(\Omega)$ respectively.

Let $K\ge 1$ be a constant. Then a homeomorphism $f\colon\Omega\to\mathbf{R}^n$ is K-quasiconformal if

$$f\in W^1_{n,\mathrm{loc}}(\Omega)$$

and

$$\bigl|Df(x)\bigr|^n\le KJ_f(x)$$

for almost every $x\in\Omega$. The references [V1] and [BI] give comprehensive introductions to quasiconformal mappings.

A Banach space A is a *real Banach algebra* if A is an algebra over $\mathbf{R}$ with norm $\|\cdot\|$ satisfying

$$\|uw\|\le\|u\|\,\|w\|$$

for each $u,w\in A$. If I is a subset of a commutative Banach algebra A, then I is an *ideal* if $uw\in I$ whenever $u\in A$ and $w\in I$. An ideal is *proper* if it is a proper subset of A, and it is *maximal* if no proper ideal contains it.

If A is a real Banach algebra, the space A' of all real-valued bounded linear functionals on A is called the *dual space* of A. If A_1 and A_2 are Banach algebras with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, and $T\colon A_1\to A_2$ is a linear transformation, the norm of T is given by

$$\|T\|=\sup\bigl\{\|Tv\|_2 : v\in A_1, \|v\|_1\le 1\bigr\}.$$

The transformation T is continuous if and only if $\|T\| < \infty$.

We let $A(\Omega)$ denote the *Royden algebra* (*over* $\mathbf{R}$) *on* Ω. This algebra consists of all real functions

$$u \in C(\Omega) \cap L^{\infty}(\Omega) \cap L_n^1(\Omega),$$

with multiplication and addition defined pointwise on Ω, and norm

$$\|u\| = \|u\|_{\Omega} = \|u\|_{\infty} + \|\nabla u\|_{n,\Omega},$$

where

$$\|\nabla u\|_{n,\Omega} = \left(\int_{\Omega} |\nabla u|^n \, dx\right)^{1/n}.$$

The algebra $A(\Omega)$ is a commutative Banach algebra that is *regular* on Ω, i.e., for each closed set $F \subset \Omega$ and each $x \in \Omega \setminus F$, there is a $u \in A(\Omega)$ such that $u(x) = 1$ and $u|F \equiv 0$. As a result, $A(\Omega)$ separates points in Ω. Furthermore, $A(\Omega)$ is *inverse-closed*, which means that if $u \in A(\Omega)$ and $\inf\{u(x) : x \in \Omega\} > 0$ then $(1/u) \in A(\Omega)$. The papers [N], [L], [L-F] and [S2] contain detailed discussions of the Royden algebra. The following theorem summarizes the relationship between quasiconformal mappings and Royden algebras.

Theorem 1.1 [L, Theorems 3.2, 7.1], [L-F, Theorem 11.3]. *Let Ω and Ω' be domains in $\mathbf{R}^n$, $n \geq 2$. Then if $f\colon \Omega \to \Omega'$ is K-quasiconformal, the transformation $f^*\colon A(\Omega') \to A(\Omega)$ defined by $f^*v = v \circ f$ for each $v \in A(\Omega')$ is an algebra isomorphism with $\|f^*\|^n \leq K$. Conversely, if $T\colon A(\Omega') \to A(\Omega)$ is an algebra isomorphism, then T induces a $\|T\|^n$-quasiconformal mapping $f\colon \Omega \to \Omega'$ such that $f^* = T$.*

Interpolating sequences will be useful for studying the Royden boundary. A sequence $\{x_i\}$ of points in Ω is an interpolating sequence for $A(\Omega)$ if for each bounded sequence of real numbers $\{r_i\}$ there exists $u \in A$ such that $u(x_i) = r_i$ for each $i \in \mathbf{N}$.

Theorem 1.2 [S2, Theorem 1.11]. *Let $\{x_i\}$ be a sequence of distinct points in Ω such that $x_i \to \partial\Omega$, then $\{x_i\}$ is an interpolating sequence for $A(\Omega)$.*

2. The Royden compactification

The *Royden compactification* Ω^* is the collection of all non-zero, bounded linear functionals χ on $A(\Omega)$ satisfying $\chi(uw) = \chi(u)\chi(w)$ for each $u, w \in A(\Omega)$. In other words, $\chi\colon A(\Omega) \to \mathbf{R}$ is a continuous homomorphism, so $\Omega^* \subset A(\Omega)'$. The original construction of the Royden compactification is in [N], and some of its important properties are outlined in [S2]. It is not hard to show that $\|\chi\| = 1$ for each $\chi \in \Omega^*$, and that Ω^* is a compact Hausdorff space in the relative weak*

topology generated by $A(\Omega)$. In this topology, if $\{\chi_\alpha\}_{\alpha\in\Lambda}$ is a net in Ω^*, then $\chi_\alpha \to \chi \in \Omega^*$ if and only if

$$\lim_\alpha |\chi_\alpha(u) - \chi(u)| = 0$$

for each $u \in A(\Omega)$.

Definition 2.1. If $x \in \Omega$, then $\hat{x} \in \Omega^*$ denotes the *point evaluation* homomorphism defined by $\hat{x}(u) = u(x)$ for each $u \in A(\Omega)$.

Because $A(\Omega)$ separates the points in Ω, the mapping $x \mapsto \hat{x}$ is a homeomorphism of Ω onto its image $\hat{\Omega} \subset \Omega^*$. By identifying each x with $\hat{x}$, it is possible to consider Ω a subset of Ω^*, and we use this convention when convenient. Because $\hat{\Omega}$ is dense in Ω^* in the weak* topology [L, p. 489], the set $\Delta = \Delta_\Omega = \Omega^* \setminus \hat{\Omega}$ constitutes a boundary known as the *Royden ideal boundary* of Ω.

Definition 2.2. If $u \in A$, then $\hat{u} \in C(\Omega^*)$ denotes the mapping defined by $\hat{u}(\chi) = \chi(u)$ for each $\chi \in \Omega^*$. The mapping $u \mapsto \hat{u}$ is an algebra homomorphism of $A(\Omega)$ onto a set $\hat{A} \subset C(\Omega^*)$.

By standard methods of functional analysis, it is possible to show that Ω^* can be identified with $\mathscr{M}$, the space of proper maximal ideals of $A(\Omega)$. If $\chi \in \Omega^*$, the explicit correspondence $M_\chi \leftrightarrow \chi$ is given by

$$M_\chi = \{u \in A : \hat{u}(\chi) = 0\} = \chi^{-1}(0) \in \mathscr{M},$$

see [S2, p. 1154].

If $T = f^*$ is the Royden algebra isomorphism described in Theorem 1.1, the *adjoint* T^* of T defined by $T^*\chi = \chi \circ T$ is a mapping of Ω into Ω'. Lewis showed that T^* is in fact a homeomorphism.

Theorem 2.3 [L, p. 490]. *If $f\colon \Omega \to \Omega'$ is quasiconformal and $T\colon A(\Omega') \to A(\Omega)$ is the corresponding Royden algebra isomorphism, then $T^*\colon \Omega^* \to \Omega'^*$ is a homeomorphism such that $T^*(\Delta) = \Delta'$ and $T^*|\Omega = f$.*

Certain ideals in $A(\Omega)$ are useful in establishing a connection between the compactification Ω^* and the algebra $A(\Omega)$. For example, if $\{x_i\}$ is a sequence in Ω then $I(x_i)$ denotes the ideal of functions tending to 0 on $\{x_i\}$, i.e.,

$$I(x_i) = I(\{x_i\}) = \{u \in A(\Omega) : \lim_{i\to\infty} u(x_i) = 0\}.$$

The fact that $A(\Omega)$ is inverse-closed easily implies the following lemma.

Lemma 2.4 [S1, Lemma 5.1]. *If $u \in A(\Omega)$, then u is contained in a proper ideal of $A(\Omega)$ if and only if $u \in I(x_i)$ for some sequence $\{x_i\}$ in Ω.*

Definition 2.5. Two sequences $\{x_i\}$ and $\{z_j\}$ in Ω are *disjoint* if there is no sequence that is a subsequence of both, i.e., if there exists an integer m such that

$$\{x_i : i \geq m\} \cap \{z_j : j \geq m\} = \emptyset.$$

Definition 2.6. A subsequence $\{x_{k(i)}\}$ of $\{x_i\}$ is a *proper subsequence* if $\{x_i\}$ has a subsequence disjoint from $\{x_{k(i)}\}$.

Lemma 2.7. *Let $\{x_i\}$ be a sequence in Ω and $\{x_{k(i)}\}$ a subsequence of $\{x_i\}$. Then $I(x_i) \subset I(x_{k(i)})$. If $x_i \to x \in \Omega$, then $I(x_i) = I(x_{k(i)})$. If $x_i \to \partial\Omega$, then the inclusion $I(x_i) \subset I(x_{k(i)})$ is proper whenever $\{x_{k(i)}\}$ is a proper subsequence of $\{x_i\}$.*

Proof. $I(x_i) \subset I(x_{k(i)})$ because $u(x_{k(i)}) \to 0$ whenever $u(x_i) \to 0$. If $x_i \to x \in \Omega$ then continuity of u at x implies $u(x_i) \to 0$ whenever $u(x_{k(i)}) \to 0$, and so $I(x_{k(i)}) \subset I(x_i)$.

Suppose $x_i \to \partial\Omega$ and $\{x_{k(i)}\}$ is a proper subsequence of $\{x_i\}$. We may assume $\{x_{k(i)}\}$ is a sequence of distinct points without altering the contents of $I(x_{k(i)})$. By definition there is a subsequence $\{x_{m(i)}\}$ of $\{x_i\}$ disjoint from $\{x_{k(i)}\}$, and we may assume $\{x_{m(i)}\}$ is a sequence of distinct points that does not intersect $\{x_{k(i)}\}$. Let $\{y_i\}$ denote the alternating sequence

$$\{y_i\} = \big\{x_{k(1)}, x_{m(1)}, x_{k(2)}, x_{m(2)}, \dots\big\}.$$

Since $y_i \to \partial\Omega$ as $i \to \infty$, Theorem 1.2 implies $\{y_i\}$ is an interpolating sequence. Therefore there exists $u \in A(\Omega)$ such that $u(y_{2j+1}) = 0$ and $u(y_{2j}) = 1$ for each $j \in \mathbf{N}$. Thus $u \in I(x_{k(i)})$ while $u \notin I(x_i)$. □

Theorem 2.8. *If $\{x_i\}$ is a sequence in Ω, then $I(x_i)$ is a maximal ideal if and only if $x_i \to x \in \Omega$.*

Proof. Sufficiency is straightforward. If $x_i \to x \in \Omega$, then $I(x_i) = \hat{x}^{-1}(0)$, a maximal ideal in $A(\Omega)$.

Necessity follows by proving the contrapositive. If it is not true that $x_i \to x \in \Omega$, then either $\{x_i\}$ has a subsequence $x_{k(i)} \to \partial\Omega$ or $\{x_i\}$ has distinct cluster points x_1, x_2 in Ω. In the first case, $\{x_i\}$ has infinitely many distinct points and so has a proper subsequence. Lemma 2.7 then implies $I(x_i)$ is not maximal. In the second case, $\{x_i\}$ has disjoint subsequences $\{x_{k(i)}\}$ and $\{x_{m(i)}\}$ tending respectively to distinct points x_1 and x_2 in Ω. Lemma 2.7 implies $I(x_i) \subset I(x_{k(i)})$. But because $A(\Omega)$ separates points in Ω, there exists $u \in A(\Omega)$ such that $u(x_1) = 0$ and $u(x_2) = 1$. Thus $u \notin I(x_i)$ while $u \in I(x_{k(i)})$, and so the inclusion $I(x_i) \subset I(x_{k(i)})$ is proper. □

We let $A_0(\Omega)$ denote the ideal of functions that tend to 0 at $\partial\Omega$. To be precise,

$$\begin{aligned} A_0(\Omega) &= \big\{u \in A(\Omega) : \text{ if } x_i \to \partial\Omega \text{ then } \lim_{i\to\infty} u(x_i) = 0\big\} \\ &= \big\{u \in C(\overline{\Omega}) \cap A(\Omega) : u|\partial\Omega \equiv 0\big\}. \end{aligned}$$

Clearly, $A_0(\Omega)$ properly contains the ideal of functions with compact support in Ω. Elements in the Royden boundary Δ can be characterized either by the fact that they annihilate $A_0(\Omega)$ or by the "size" of their neighborhood bases in the weak* topology.

Theorem 2.9 [L, Lemma 6.2], [S2, Theorem 4.7]. *The following statements are equivalent:*

1) $\chi \in \Delta$.
2) χ *has no countable neighborhood basis in* Ω^*.
3) $\chi(u) = 0$ *for each* $u \in A_0(\Omega)$.
4) $\chi(u) = 0$ *for each* $u \in A$ *with compact support in* Ω.

3. Elements of the Royden boundary characterized as nets

Because elements of Δ have no countable neighborhood bases, we cannot adequately describe convergence in Ω^* to elements of Δ using the theory of sequences. We must rely on the more general but less intuitive theory of nets, which is described well in [K]. Briefly, if Λ is an indexing set directed by a binary relation $\geq$, then $\{x_\alpha\}_{\alpha\in\Lambda}$ is a net in a set E if $x_\alpha \in E$ for each $\alpha \in \Lambda$. The net $\{x_\alpha\}$ is *eventually* in a set U if there exists $\beta \in \Lambda$ such that $\alpha \geq \beta$ implies $x_\alpha \in U$; it is *frequently* in U if for each $\alpha \in \Lambda$ there is a $\beta \in \Lambda$ such that $\beta \geq \alpha$ and $x_\beta \in U$. A net in a topological space X converges to $x \in X$ if and only if it is eventually in each neighborhood of x, in which case x is a *limit point* of the net and we write $\lim_\alpha x_\alpha = x$. A point $s \in X$ is a *cluster point* of the net if the net is frequently in each neighborhood of s. A net $\{y_\beta\}_{\beta\in B}$ is a subnet of $\{x_\alpha\}_{\alpha\in\Lambda}$ if and only if there exists a function g defined on B with values $g(\beta) \in \Lambda$ such that $y_\beta = x_{g(\beta)}$ for each $\beta \in B$, and for each $\alpha \in \Lambda$ there exists $\gamma \in B$ such that if $\beta \geq \gamma$ then $g(\beta) \geq \alpha$. A net $\{x_\alpha\}$ has s as a cluster point if and only if some subnet of $\{x_\alpha\}$ converges to s.

The theory of nets allows us to show that elements of Δ can be characterized as special kinds of nets for which no subnet is a sequence. Before presenting this characterization, however, we define a set of functions that will be useful in proving several results. For each $z \in \mathbf{R}^n$, we define a "cutoff" of the distance function:

$$\sigma_z(x) = \begin{cases} |x-z|, & x \in \mathbf{B}(z,1) \\ 1, & x \in \mathbf{R}^n \setminus \mathbf{B}(z,1). \end{cases}$$

For $z = \infty$, we define

$$\sigma_\infty(x) = \frac{1}{1+|x|}.$$

These functions have the properties that $\sigma_z(x) \to 0$ whenever $x \to z$, $\sigma_z(x) = 0$ if and only if $x = z$, and $\sigma_z(x) \leq 1$ for each $x \in \mathbf{R}^n$. Furthermore, straightforward calculations show that $\|\nabla\sigma_z\|_{n,\Omega} < \infty$, so $\sigma_z \in A(\mathbf{R}^n)$ and $\sigma_z|\Omega \in A(\Omega)$ for each $z \in \overline{\mathbf{R}}^n$. In general, we will let σ_z denote $\sigma_z|\Omega$ which, of course, extends continuously to $\overline{\Omega}$.

Theorem 3.1. *Let Ω be a domain in $\mathbf{R}^n$. Then $\chi \in \Delta$ if and only if there exists a net $\{z_\alpha\}_{\alpha\in\Lambda}$ in Ω such that*

1) *$\hat{z}_\alpha \to \chi$ in Ω^*,*
2) *there exists $z \in \partial\Omega$ for which $z_\alpha \to z$ in $\overline{\Omega}$,*
3) *$\{z_\alpha\}$ has no subnet that is a sequence.*

Proof. First we show necessity. If $\chi \in \Delta$, then because $\hat{\Omega}$ is dense in Ω^*, there is a net $\{\hat{z}_\alpha\}$ in $\hat{\Omega}$ such that $\hat{z}_\alpha \to \chi$ in Ω^*. Thus (1) holds.

For (2), $\overline{\Omega}$ is compact in $\overline{\mathbf{R}}^n$, and so $\{z_\alpha\}$ has a cluster point $z \in \overline{\Omega}$ [K, p. 136]. If $\{z_\alpha\}$ does not converge to z, then there is a subnet of $\{z_\alpha\}$ which has no subnet converging to z [K, p. 74]. Because $\overline{\Omega}$ is compact, this subnet has a cluster point $y \neq z$. Thus $\{z_\alpha\}$ has a subnet on which the function σ_z tends to $\sigma_z(y) > 0$ and a subnet on which σ_z tends to 0; in other words, $\lim_\alpha \hat{z}_\alpha(\sigma_z)$ does not exist. This means $\{\hat{z}_\alpha\}$ does not converge in Ω^*, a contradiction to (1). Clearly $z \in \partial\Omega$; for if not, $\hat{z}_\alpha \to \hat{z} \in \hat{\Omega}$, a contradiction to $\chi \in \Delta$.

For (3), suppose $\{z_\alpha\}$ has a subnet $\{z_i\}$ that is a sequence. Then $z_i \to z$ [K, p. 74], and Theorem 1.2 implies $\{z_i\}$ has a subsequence of distinct points that is an interpolating sequence. However, $\{\hat{z}_i\}$ is a subnet of $\{\hat{z}_\alpha\}$ and thus $\hat{z}_i \to \chi$ in Ω^*. This means

$$\lim_{i\to\infty} u(z_i) = \chi(u)$$

for each $u \in A(\Omega)$, a contradiction to $\{z_i\}$ having an interpolating subsequence.

For sufficiency, assume (1), (2), and (3) hold, and suppose $\chi = \hat{x} \in \hat{\Omega}$. Property (2) implies $\lim_\alpha \sigma_z(z_\alpha) = 0$, and so (1) implies $\hat{x}(\sigma_z) = 0$. However,

$$\hat{x}(\sigma_z) = \sigma_z(x) > 0,$$

a contradition. Thus $\chi \in \Delta$. □

An algebraic restatement of part of Theorem 3.1 yields the following corollary.

Corollary 3.2. *If $\chi \in \Delta$ and M denotes the maximal ideal $\chi^{-1}(0)$ in $A(\Omega)$, then there exists a net $\{z_\alpha\}_{\alpha\in\Lambda}$ in Ω converging to $z \in \partial\Omega$ for which no subnet is a sequence and*

$$M = \{u \in A(\Omega) : \lim_\alpha u(z_\alpha) = 0\}.$$

Definition 3.3. Let $\{z_\alpha\}$ be a net in Ω. We say $\{z_\alpha\}$ is a *Royden net* if $\hat{z}_\alpha$ converges to a point $\chi \in \Delta$. Theorem 3.1 says that each $\chi \in \Delta$ is determined by a *corresponding* Royden net. A particular χ may have many corresponding Royden nets; however, each must converge to the same point in $\overline{\Omega}$.

Theorem 3.4. *If $\{z_\alpha\}$ and $\{z_\beta\}$ are nets in Ω such that $\hat{z}_\alpha$ and $\hat{z}_\beta$ converge in Ω^* to a common point $\chi \in \Delta$, then z_α and z_β converge in $\overline{\Omega}$ to a common point $z \in \partial\Omega$.*

Proof. By Theorem 3.1, $z_\alpha \to z \in \partial\Omega$ and $z_\beta \to y \in \partial\Omega$. Thus,

$$0 = \lim_\alpha \hat{z}_\alpha(\sigma_z) = \chi(\sigma_z) = \lim_\beta \hat{z}_\beta(\sigma_z) = \lim_\beta \sigma_z(z_\beta) = \sigma_z(y),$$

and hence $z = y$. □

4. Fibers in the Royden boundary

By definition, each element in $\hat{\Omega}$ corresponds to a unique point in Ω, and we have shown that each element in Δ corresponds to nets in Ω that must converge in the Euclidean topology to a unique point in $\partial\Omega$. These correspondences can be expressed explicitly by a projection mapping $\pi\colon \Omega^* \to \overline{\Omega}$. Let $\chi \in \Omega^*$; if $\chi = \hat{x} \in \hat{\Omega}$, then

$$\pi(\chi) = x;$$

if $\chi \in \Delta$, then

$$\pi(\chi) = \text{ the limit point } z \in \partial\Omega \text{ of a Royden net corresponding to } \chi.$$

Theorem 3.4 shows that π is well-defined, and it is not hard to see that $\pi|\hat{\Omega}$ is a bijective mapping of $\hat{\Omega}$ onto Ω with $\pi^{-1}(x) = \hat{x}$ for each $x \in \Omega$.

Lemma 4.1. *If u is continuous on $\overline{\Omega}$, then $\chi(u) = u(\pi(\chi))$.*

Proof. If $\chi = \hat{x} \in \hat{\Omega}$, then $\pi(\chi) = x$, so that $\hat{x}(u) = u(x) = u(\pi(\chi))$. If $\chi \in \Delta$, Theorem 3.1 implies there is a corresponding Royden net $\{z_\alpha\}$ in Ω such that $\hat{z}_\alpha \to \chi$ and $z_\alpha \to z \in \partial\Omega$. Then $\chi = \lim_\alpha \hat{z}_\alpha$ and continuity of u implies

$$\chi(u) = \lim_\alpha \hat{z}_\alpha(u) = \lim_\alpha u(z_\alpha) = u(z) = u(\pi(\chi)). \; □$$

Theorem 4.2. *The projection mapping π is continuous.*

Proof. We must show that if $\chi_\alpha \to \chi$ in Ω^*, then $\pi(\chi_\alpha) \to \pi(\chi)$ in $\overline{\Omega}$. Let $\chi_\alpha \to \chi$; then the definition of convergence in the weak* topology yields

$$\lim_\alpha \left|\chi_\alpha(\sigma_{\pi(\chi)}) - \chi(\sigma_{\pi(\chi)})\right| = 0.$$

Lemma 4.1 implies

$$\chi(\sigma_{\pi(\chi)}) = \sigma_{\pi(\chi)}(\pi(\chi)) = 0;$$

therefore $\lim_\alpha \chi_\alpha(\sigma_{\pi(\chi)}) = 0$, which, by Lemma 4.1, implies $\lim_\alpha \sigma_{\pi(\chi)}(\pi(\chi_\alpha)) = 0$. The definition of $\sigma_{\pi(\chi)}$ then implies $\pi(\chi_\alpha) \to \pi(\chi)$. □

Theorem 4.3. *$\pi|\Delta$ is a surjective mapping of Δ onto $\partial\Omega$.*

Proof. Let $\chi \in \Delta$ with $\{z_\alpha\}$ a corresponding Royden net such that $z_\alpha \to z \in \partial\Omega$. By definition, $\pi(\chi) = z$, and so $\pi(\Delta) \subset \partial\Omega$. Furthermore, the fact that $\pi(\hat{\Omega}) = \Omega$ implies $\Omega \subset \pi(\Omega^*) \subset \overline{\Omega}$. Because Ω^* is compact, the continuity of π implies $\pi(\Omega^*)$ is a compact subset of $\overline{\Omega}$. But the only compact subset of $\overline{\Omega}$ that contains Ω is the set $\overline{\Omega}$ itself. Hence $\pi(\Omega^*) = \overline{\Omega}$ and $\pi(\Delta) = \overline{\Omega} \setminus \Omega = \partial\Omega$. □

Definition 4.4. If $z \in \overline{\Omega}$, then Φ_z denotes the *fiber over* z, defined as

$$\Phi_z = \pi^{-1}(z) = \bigl\{\chi \in \Omega^* : \chi \text{ corresponds to a Royden net converging to } z\bigr\}.$$

Continuity of π implies Φ_z is a closed subset of Δ and hence compact. Also $\Phi_z \cap \Phi_y = \emptyset$ whenever $z \neq y$, and Theorem 4.3 implies $\Delta = \cup\{\Phi_z : z \in \partial\Omega\}$.

For bounded domains there is an equivalent definition of π. If Ω is bounded, then p_i, the ith coordinate projection mapping in $\mathbf{R}^n$ restricted to Ω, belongs to $A(\Omega)$. We may define π by

$$\pi(\chi) = \bigl(\chi(p_1), \ldots, \chi(p_n)\bigr) = \bigl(\hat{p}_1(\chi), \ldots, \hat{p}_n(\chi)\bigr) \in \mathbf{R}^n.$$

For example, if $\hat{x} \in \hat{\Omega}$, then

$$\pi(\hat{x}) = \bigl(\hat{x}(p_1), \ldots, \hat{x}(p_n)\bigr) = (x_1, \ldots, x_n) = x;$$

and if $\chi \in \Delta$, and $\{z_\alpha\}$ is a Royden net corresponding to χ, then

$$\pi(\chi) = \bigl(\lim_\alpha \hat{z}_\alpha(p_1), \ldots, \lim_\alpha \hat{z}_\alpha(p_n)\bigr) = \bigl(\lim_\alpha p_1(z_\alpha), \ldots, \lim_\alpha p_n(z_\alpha)\bigr) = z \in \partial\Omega.$$

Lemma 4.5. *$\chi \in \Phi_z$ if and only if $\chi(\sigma_z) = 0$.*

Proof. Suppose $\chi \in \Phi_z$ with corresponding Royden net $\{z_\alpha\}$. Then $\chi = \lim_\alpha \hat{z}_\alpha$, so

$$\chi(\sigma_z) = \lim_\alpha \hat{z}_\alpha(\sigma_z) = \lim_\alpha \sigma_z(z_\alpha) = 0.$$

Conversely, suppose $\chi(\sigma_z) = 0$. Then Lemma 4.1 implies $\sigma_z\bigl(\pi(\chi)\bigr) = 0$. But σ_z vanishes only at z, so $\pi(\chi) = z$, i.e. $\chi \in \Phi_z$. □

Theorem 4.6. *Let $u \in A(\Omega)$, $z \in \partial\Omega$, and $r \in \mathbf{R}$. There exists $\chi \in \Phi_z$ for which $\chi(u) = r$ if and only if there exists a sequence $\{z_j\}$ in Ω such that $z_j \to z$ and $u(z_j) \to r$.*

Proof. Let $\chi \in \Phi_z$ and $\chi(u) = r$. Let $\{z_\alpha\}$ be a Royden net corresponding to χ. Suppose there is no sequence $\{z_j\}$ in Ω for which $z_j \to z$ and $u(z_j) \to r$. Then there exists $\varepsilon > 0$ such that

$$|u(x) - r| \geq c > 0$$

for each $x \in \mathbf{B}(z, \varepsilon) \cap \Omega$. Consequently $|\hat{z}_\alpha(u) - r| \geq c$ for each $z_\alpha \in \mathbf{B}(z, \varepsilon) \cap \Omega$, and hence $\chi(u) \neq r$, a contradiction.

Conversely, let $z_j \to z$ and $u(z_j) \to r$. Then by Zorn's lemma, there is a maximal ideal M containing $I(z_j)$; and by the Gelfand identification, $M = \chi^{-1}(0)$ for some $\chi \in \Omega^*$. But $\sigma_z \in I(z_j) \subset M$, and so $\chi(\sigma_z) = 0$. Thus Lemma 4.5 implies $\chi \in \Phi_z$. Furthermore, $u - r \in I(z_j)$, so $\chi(u - r) = 0$, and hence $\chi(u) = r$. □

Corollary 4.7. *If $u \in A(\Omega)$ and $z \in \partial\Omega$, then the range of $\hat{u}$ on Φ_z is exactly the set of all real numbers r for which there exists a sequence $\{z_j\}$ in Ω with $z_j \to z$ and $u(z_j) \to r$.*

Proof. Let r lie in the range of $\hat{u}$ on Φ_z. Then there exists $\chi \in \Phi_z$ such that $\chi(u) = \hat{u}(\chi) = r$. So Theorem 4.6 implies there exists a sequence $\{z_j\}$ in Ω such that $z_j \to z$ and $u(z_j) \to r$. Conversely, let $\{z_j\}$ be a sequence in Ω such that $z_j \to z$ and $u(z_j) \to r$. By Theorem 4.6, there exists $\chi \in \Phi_z$ for which $r = \chi(u) = \hat{u}(\chi)$; hence r is in the range of $\hat{u}$ on Φ_z. □

If $u \in A(\Omega)$, we say u has a limit at $z \in \partial\Omega$ if there is some $r \in \mathbf{R}$ such that u converges to r on every sequence in Ω converging to z. A special case of Corollary 4.7 is the following.

Corollary 4.8. *If $u \in A(\Omega)$ and $z \in \partial\Omega$, then u has a limit r at z if and only if $\chi(u) = r$ for each $\chi \in \Phi_z$.*

Proof. Let u have a limit r at z. Then u converges to r on every sequence in Ω converging to z. By Corollary 4.7, $\{r\}$ is the range of $\hat{u}$ on Φ_z, and so $r = \hat{u}(\chi) = \chi(u)$ for each $\chi \in \Phi_z$. Conversely, let r be such that $r = \chi(u) = \hat{u}(\chi)$ for each $\chi \in \Phi_z$. If $\{z_j\}$ is a sequence in Ω converging to z, then Corollary 4.7 implies each convergent subsequence of $\{u(z_j)\}$ converges to r. But $\{u(z_j)\}$ is bounded, and so this means $\{u(z_j)\}$ converges to r. □

It is difficult to rely on our intuition about sequences to fully understand the behavior of nets. This is illustrated explicitly in the following corollary. Sequences tending to $\partial\Omega$ in Ω are interpolating sequences for $A(\Omega)$; however, an analog of this fact for nets does not hold.

Corollary 4.9. *For each $z \in \partial\Omega$ there is a net $\{z_\alpha\}$ in Ω such that $z_\alpha \to z$ in the Euclidean topology and $\lim_\alpha u(z_\alpha)$ exists for each $u \in A(\Omega)$.*

Proof. The surjectivity of π implies Φ_z is not empty. If $\chi \in \Phi_z$ and $\{z_\alpha\}$ is a corresponding Royden net, then $\chi = \lim_\alpha \hat{z}_\alpha$ in Ω^* and the definition of convergence in the weak* topology implies

$$\lim_\alpha \hat{z}_\alpha(u) = \lim_\alpha u(z_\alpha) = \chi(u)$$

for each $u \in A(\Omega)$. □

5. Homeomorphic boundary extensions of quasiconformal mappings

Because each boundary fiber in a domain's Royden compactification corresponds explicitly to a particular boundary point, it is possible to use the concept of boundary fibers to characterize the existence of homeomorphic boundary extensions of quasiconformal mappings. Such extensions exist if and only if the induced homeomorphism between Royden compactifications maps boundary fibers onto boundary fibers.

Theorem 5.1. *Let $f\colon \Omega \to \Omega'$ be a quasiconformal mapping and $T = f^*\colon A(\Omega') \to A(\Omega)$ be its corresponding Royden algebra isomorphism. Then f has a homeomorphic extension $\overline{f}\colon \overline{\Omega} \to \overline{\Omega}'$ if and only if for each $z \in \partial\Omega$ there exists $y \in \partial\Omega'$ such that $T^*(\Phi_z) = \Phi_y$, in which case $y = \overline{f}(z)$.*

Proof. To prove necessity, observe that each $\chi \in \Phi_z$ has a corresponding Royden net $\{z_\alpha\}$. Let $y_\alpha = \overline{f}(z_\alpha)$ and $y = \lim_\alpha y_\alpha = \overline{f}(z)$. Since $T^*\hat{x} = \widehat{f(x)}$ for each $x \in \Omega$,

$$T^*\chi = T^*(\lim_\alpha \hat{z}_\alpha) = \lim_\alpha T^*\hat{z}_\alpha = \lim_\alpha \hat{y}_\alpha.$$

Thus $T^*\chi$ has $\{y_\alpha\}$ as a corresponding Royden net, i.e. $T^*\chi \in \Phi_y$. So $T^*(\Phi_z) \subset \Phi_y$ and a similar argument shows ${T^*}^{-1}(\Phi_y) \subset \Phi_z$. Hence $T^*(\Phi_z) = \Phi_y$.

To prove sufficiency, for each $z \in \partial\Omega$ define $\overline{f}(z)$ to be that point $y \in \partial\Omega$ for which $T^*(\Phi_z) = \Phi_y$. We first show $\overline{f}$ is injective by showing that if z and x are distinct points in $\partial\Omega$, then $T^*(\Phi_z) \neq T^*(\Phi_x)$. Let $v = T^{-1}\sigma_z$. By Lemma 4.5, $\chi \in \Phi_z$ implies $\chi(\sigma_z) = 0$ and $\eta \in \Phi_x$ implies $\eta(\sigma_z) > 0$. If $\chi' \in T^*(\Phi_z)$, then there exists $\chi \in \Phi_z$ for which $T^*\chi = \chi'$. It follows that

$$\chi'(v) = T^*\chi(v) = \chi(Tv) = \chi(\sigma_z) = 0.$$

Similarly, if $\eta' \in T^*(\Phi_x)$, then $\eta'(v) = \eta(\sigma_z) > 0$. Hence $T^*(\Phi_z) \neq T^*(\Phi_x)$.

We next show $\overline{f}$ is surjective. Let $y \in \partial\Omega'$. Then there is a sequence $\{y_j\}$ in Ω' for which $y_j \to y$. Because f is a homeomorphism, $f^{-1}(y_j) \to \partial\Omega$, and because $\overline{\Omega}$ is compact, there is $z \in \partial\Omega$ for which a subseqence $\{z_k\}$ of $\{f^{-1}(y_j)\}$ converges to z. By hypothesis, there is $w \in \partial\Omega'$ for which $T^*\Phi_z = \Phi_w$. We show $w = y$.

Let $u = T\sigma_w$. The continuity of σ_w at w and Corollary 4.8 imply that for each $\chi \in \Phi_z$

$$\chi(u) = T^*\chi(\sigma_w) = 0.$$

This means, again by Corollary 4.8, that u has the limit 0 at z, and so $u(z_k) \to 0$. But

$$u(z_k) = T\sigma_w(z_k) = f^*\sigma_w(z_k) = \sigma_w\big(f(z_k)\big),$$

and so $\sigma_w\big(f(z_k)\big) \to 0$. Because $f(z_k) \to y$, the fact that σ_w vanishes only at w implies $w = y$.

Finally, we show $\overline{f}$ and $\overline{f}^{-1}$ are continuous. Because f and f^{-1} are homeomorphisms, it suffices to show $\overline{f}$ is continuous on $\partial\Omega$. A similar argument shows $\overline{f}^{-1}$ is continuous on $\partial\Omega'$. Let z_j converge in $\overline{\Omega}$ to $z \in \partial\Omega$. We must show $\overline{f}(z_j)$ converges in $\overline{\Omega}'$ to $\overline{f}(z)$.

Suppose this is not true. Then because $\overline{\Omega}'$ is compact, $\{\overline{f}(z_j)\}$ has a subsequence $\{y_k\}$ converging to $y \neq \overline{f}(z)$, and because f is a homeomorphism, $y \in \partial\Omega'$. Let z_k denote the sequence $\{\overline{f}^{-1}(y_k)\}$. Because $\overline{f}$ is surjective, there exists $x \in \partial\Omega$ for which $T^*(\Phi_x) = \Phi_y$. Clearly $x \neq z$, because $T^*(\Phi_z) = \Phi_{\overline{f}(z)} \neq \Phi_y$. If $v = T^{-1}\sigma_z$, then for each $\chi \in \Phi_x$

$$T^*\chi(v) = \chi(\sigma_z) = \sigma_z(x) > 0.$$

Because T^* maps Φ_x onto Φ_y, this inequality means $\eta(v) > 0$ for each $\eta \in \Phi_y$.

We claim the existence of a sequence $\{w_k\}$ in Ω' such that

$$|w_k - y_k| < \frac{1}{k} \qquad \text{and} \qquad |v(w_k)| < 2\sigma_z(z_k)$$

for each k. Then $w_k \to y$; and since $\sigma_z(z_k) \to 0$, it follows that $v(w_k) \to 0$ as $k \to \infty$. By Theorem 4.6, there exists $\eta \in \Phi_y$ for which $\eta(v) = 0$, a contradiction. Therefore, our argument is complete if we prove the claim.

We construct $\{w_k\}$ in two steps. First, if $y_k \in \Omega'$, let $w_k = y_k$. Then $|w_k - y_k| = 0$ and

$$v(w_k) = T^{-1}\sigma_z(y_k) = (f^{-1})^*\sigma_z(y_k) = \sigma_z(z_k).$$

Second, if $y_k \in \partial\Omega'$, then for each $\xi \in \Phi_{y_k}$ there is $\chi \in \Phi_{z_k}$ such that $T^*\chi = \xi$. Hence,

$$\xi(v) = T^*\chi(v) = \chi(\sigma_z) = \sigma_z(z_k)$$

for each $\xi \in \Phi_{y_k}$. Corollary 4.8 now implies that v has the limit $\sigma_z(z_k)$ at y_k. Thus, it is possible to choose $w_k \in \Omega'$ close enough to y_k to satisfy the two conditions in the claim. □

6. Boundary fibers and homogeneity

As a result of Theorem 5.1, it is possible to show that all boundary fibers of domains satisfying certain homogeneity conditions on the boundary are homeomorphic to each other. To describe this condition precisely, we define the concept of transitivity. If Γ is a family of homeomorphisms of a set X onto itself then Γ *acts transitively* on a set $E \subset X$ if for each $a, b \in E$ there exists $g \in \Gamma$ such that $g(a) = b$. Let $\Gamma(\Omega)$ denote the family of self-homeomorphisms $f\colon \overline{\Omega} \to \overline{\Omega}$ such that $f|\Omega$ is quasiconformal. In this section, we consider the class of domains for which $\Gamma(\Omega)$ acts transitively on $\partial\Omega$. Using terminology from Gehring and Palka [GP], we might say that such domains are *quasiconformally homogeneous with respect to the boundary*. A typical domain of this type is $\mathbf{B}^n$, on which the family of rotations of $\mathbf{B}^n$ acts transitively. We will show, however, that even for $\mathbf{B}^n$, the Royden boundary $\Delta\mathbf{B}^n$ is not the natural product of its boundary $\mathbf{S}^{n-1}$ with any particular boundary fiber.

Theorem 6.1. *Let $\Gamma(\Omega)$ act transitively on $\partial\Omega$. If $y, z \in \partial\Omega$, then Φ_z is homeomorphic to Φ_y.*

Proof. Let $f \in \Gamma(\Omega)$ be such that $f(z) = y$. Then $f|\Omega$ is quasiconformal,

$$(f|\Omega)^* = T\colon A(\Omega) \to A(\Omega)$$

is a Royden algebra isomorphism, and $T^*\colon \Omega^* \to \Omega^*$ is a homeomorphism. We only need to show that $T^*(\Phi_z) \subset \Phi_y$, for then Theorem 5.1 implies $T^*(\Phi_z) = \Phi_y$ and $T^*|\Phi_z$ is the desired homeomorphism.

Let $\chi \in \Phi_z$ with corresponding Royden net $\{z_\alpha\}$ and $f(z_\alpha) = y_\alpha \to y$. Then

$$T^*\chi = T^*(\lim_\alpha \hat{z}_\alpha) = \lim_\alpha T^*\hat{z}_\alpha = \lim_\alpha \hat{y}_\alpha,$$

so $\{y_\alpha\}$ is a Royden net for $T^*\chi$, i.e. $T^*\chi \in \Phi_y$. □

Let e denote the unit vector $(1, 0, 0, \ldots, 0) \in \mathbf{S}^{n-1}$, and $\Delta = \Delta\mathbf{B}^n$. Because

$$\Delta = \cup\{\Phi_z : z \in \mathbf{S}^{n-1}\},$$

and because Δ can be identified with $\mathbf{S}^{n-1} \times \Phi_e$ by rotations of $\overline{\mathbf{B}^n}$, Theorem 6.1 leads to the question of whether or not Δ is homeomorphic to $\mathbf{S}^{n-1} \times \Phi_e$ in the natural way. In other words, if for each $z \in \mathbf{S}^{n-1}$, $T_z^*\colon \Phi_e \to \Phi_z$ denotes the homeomorphism induced by the rotation of $\overline{\mathbf{B}^n}$ mapping e to z, is the function $h\colon \mathbf{S}^{n-1} \times \Phi_e \to \Delta$ given by $h(z, \eta) = T_z^*(\eta)$ a homeomorphism? Theorem 6.4 shows the answer is no. This is a result of the nonexistence of continuous sections between $\mathbf{S}^{n-1}$ and Δ, a fact we prove in Theorem 6.3.

Lemma 6.2. *If $\{\chi_j\}$ is a convergent sequence in Δ, then $\{\chi_j\}$ is eventually in a single fiber Φ_z, $z \in \mathbf{S}^{n-1}$.*

Proof. Let $\chi_j \to \chi \in (\mathbf{B}^n)^*$, and let

$$z_j = \pi(\chi_j) \in \mathbf{S}^{n-1}$$

and $z = \pi(\chi)$. Suppose $\{z_j\}$ has a subsequence $\{z_k\}$ such that for each k, $z_k \neq z$. Then we consider two cases.

First, suppose some point $y \neq z$ appears infinitely often in the sequence $\{z_k\}$. Then the sequence $\{\chi_j(\sigma_z)\}$ assumes the value $|y - z| > 0$ an infinite number of times. However, the fact that $\chi_j \to \chi$ implies

$$\chi_j(\sigma_z) \to \chi(\sigma_z) = 0,$$

a contradiction.

Second, suppose each point in the sequence $\{z_k\}$ appears only finitely many times. Then $\{z_k\}$ contains a subsequence of distinct points, again denoted by $\{z_k\}$. Let $\{B_k\}$ be a sequence of balls in $\mathbf{R}^n$ with mutually disjoint closures centered at z_k. There exist functions $w_k \in A(\mathbf{R}^n)$ such that $w_k(z_k) = (-1)^k$, $|w_k| \le 1$, $w_k|\mathbf{R}^n \setminus B_k \equiv 0$ and

$$\left(\int_{\mathbf{R}^n} \left|\Delta w_k(x)\right|^n dx\right)^{1/n} \le 2^{-k}$$

for each $k \in \mathbf{N}$, (this follows from the fact that the conformal capacity of a spherical ring with a degenerate boundary component is 0 [G1, p. 138]). Therefore

$$v(x) = \sum_{i=1}^{\infty} w_k(x)$$

is a function in $A(\mathbf{R}^n)$. Furthermore, because v is the sum of continuous functions with disjoint support, $u = v|\Omega \in C(\Omega)$. Thus $u \in A(\Omega)$, and u can be continuously extended to each of the boundary points z_k such that

$$u(z_k) = w_k(z_k) = (-1)^k.$$

But Corollary 4.8 implies $\chi_k(u) = u(z_k)$, and so $\{\chi_k(u)\}$ is not a convergent sequence, a contradiction to the hypothesis that $\{\chi_j\}$ is convergent in $(\mathbf{B}^n)^*$. □

Theorem 6.3. *If $s\colon \mathbf{S}^{n-1} \to \Delta$ is any function for which $\pi \circ s$ is the identity on $\mathbf{S}^{n-1}$, then s cannot be continuous.*

Proof. Suppose $\pi \circ s$ is the identity on $\mathbf{S}^{n-1}$ (i.e. s is a *section*) and s is continuous. Let $\{z_j\}$ be a sequence of distinct points in $\mathbf{S}^{n-1}$ with $z_j \to z \in \mathbf{S}^{n-1}$. Then $s(z_j) \to s(z)$ in Ω^*, and Lemma 6.2 implies there exists an integer m and $y \in \mathbf{S}^{n-1}$ such that $s(z_j) \in \Phi_y$ for each $j \ge m$. Hence $z_j = \pi\bigl(s(z_j)\bigr) = y$ for each $j \ge m$, a contradiction. □

Theorem 6.4. *Δ is not homeomorphic to $\mathbf{S}^{n-1} \times \Phi_e$ in the natural way.*

Proof. Suppose the function $h\colon \mathbf{S}^{n-1} \times \Phi_e \to \Delta$ defined by $h(z,\eta) = T_z^*(\eta)$ is a homeomorphism. If η is fixed in Φ_e, then the function $s\colon \mathbf{S}^{n-1} \to \Delta$ defined by $s(z) = h(z,\eta)$ is a continuous function. But $\pi \circ s$ is the identity on $\mathbf{S}^{n-1}$ because

$$\pi \circ s(z) = \pi\bigl(T_z^*(\eta)\bigr) = z$$

for each $z \in \mathbf{S}^{n-1}$. This contradicts Theorem 6.3. □

7. Correspondence between prime ends and boundary fibers

Generalizations of the theory of homeomorphic boundary extensions lead naturally to a discussion of prime ends. Theorem 5.1 indicates a possible connection between the boundary fibers and prime ends of Ω. We make this connection explicit by showing that whenever Ω is finitely connected on the boundary, the set of components of boundary fibers and the set of prime ends are one and the same.

A domain Ω is finitely connected at $z \in \partial\Omega$ if z has arbitrarily small neighborhoods in $\mathbf{R}^n$ whose intersection with Ω has a finite number of components. If this number is one, Ω is *locally connected* at z. The domain Ω is *finitely connected on the boundary* or *locally connected on the boundary* if these conditions hold for each $z \in \partial\Omega$.

The theory of prime ends provides a method for defining connectedness of domains at their boundaries. Caratheodory initiated this theory by showing that each conformal mapping $f\colon \mathbf{B}^2 \to \Omega$ has a homeomorphic extension from $\overline{\mathbf{B}^2}$ onto a compactification $\tilde{\Omega}$ of Ω obtained by combining Ω with its set of prime ends. Näkki and Zorich applied this idea to domains in higher dimensions and obtained similar results for quasiconformal mappings of domains in n-space [Nk], [Z]. Väisälä gave a particularly simple description of prime ends for domains finitely connected on the boundary [V2]. We use his description to prove our theorem.

If Ω is a domain in $\mathbf{R}^n$, then an *endcut* of Ω is a path $\gamma\colon [a,b) \to \Omega$ such that $\gamma(t) \to z \in \partial\Omega$ as $t \to b$. We associate z to γ by the notation $z = h(\gamma)$. A subendcut of γ is a restriction of γ to a subinterval $[a_1, b)$. If U is a neighborhood of $h(\gamma)$, then there is a unique component $P(U,\gamma)$ of $U \cap \Omega$ containing a subendcut of γ. Two endcuts γ and λ are equivalent if $h(\gamma) = h(\lambda)$ and if $P(U,\gamma) = P(U,\lambda)$ for each neighborhood U of $h(\gamma)$. The equivalence class $\tilde{\gamma}$ of γ is a *prime end* of Ω, and the collection $\mathscr{P}$ of all prime ends is the *prime end boundary* of Ω. The set $\tilde{\Omega} = \mathscr{P} \cup \Omega$ is the *prime end compactification* of Ω. A natural impression map $\theta_\Omega\colon \tilde{\Omega} \to \overline{\Omega}$ is defined by $\theta_\Omega(\tilde{\gamma}) = h(\gamma)$ for $\tilde{\gamma} \in \mathscr{P}$ and $\theta_\Omega(x) = x$ for $x \in \Omega$. If Ω is locally connected at $z \in \partial\Omega$, then $\theta_\Omega^{-1}(z)$ is a single point. Thus if Ω is locally connected on the boundary, $\mathscr{P}$ can be identified with $\partial\Omega$. The following lemmas show that analogous equivalences can be defined for elements in the Royden boundary.

Lemma 7.1. *Let Ω be finitely connected on the boundary with $z \in \partial\Omega$. If $\chi \in \Phi_z$ and U is a neighborhood of z in $\overline{\mathbf{R}}^n$, then there is a unique component $Q(U,\chi)$ of $U \cap \Omega$ such that each Royden net $\{z_\alpha\}$ corresponding to χ is eventually in $Q(U,\chi)$.*

Proof. Let $\hat{z}_\alpha \to \chi$. First we claim that $\{z_\alpha\}$ is eventually in a single component of $U \cap \Omega$. If not, then since $U \cap \Omega$ has a finite number of components, there must be a component V of $U \cap \Omega$ such that $\{z_\alpha\}$ is both frequently inside and frequently outside V. We define a variant of the function σ_z by choosing n

small enough so that $\mathbf{B}(z,1/n)\subset U$ and letting

$$u(x)=\begin{cases}|x-z|, & x\in V\cap \mathbf{B}(z,1/n),\\ 1/n, & x\in \Omega\setminus\big(V\cap \mathbf{B}(z,1/n)\big).\end{cases}$$

Then $u\in A(\Omega)$, and the net $\{\hat z_\alpha(u)\}$ has a subnet converging to 0 and a subnet converging to $1/n$; hence $\{\hat z_\alpha(u)\}$ does not converge to $\chi(u)$, a contradiction.

Furthermore, let $\{z_\alpha\}$ and $\{z_\beta\}$ be distinct Royden nets corresponding to χ. If there are distinct components V and W of $U\cap\Omega$ such that $\{z_\alpha\}$ is eventually in V and $\{z_\beta\}$ is eventually in $W\subset \Omega\setminus V$, then for u defined as above, we obtain $\chi(u)=\lim_\alpha \hat z_\alpha(u)=0$ and $\chi(u)=\lim_\beta \hat z_\beta(u)=1/n$, a contradiction. □

Theorem 7.2. *Two elements χ and η in Φ_z are in the same component of Φ_z if and only if $Q(U,\chi)=Q(U,\eta)$ for each neighborhood U of z in $\overline{\mathbf{R}}^n$.*

Proof. Suppose there is a neighborhood U of z for which $Q(U,\chi)\neq Q(U,\eta)$. If n is chosen small enough so that $\mathbf{B}(z,1/n)\subset U$ and

$$u(x)=\begin{cases}|x-z|, & x\in Q(U,\chi)\cap \mathbf{B}(z,1/n),\\ 1/n, & x\in \Omega\setminus\big(Q(U,\chi)\cap \mathbf{B}(z,1/n)\big),\end{cases}$$

then $u\in A(\Omega)$, and by Corollary 4.7 the range of $\hat u$ on Φ_z is the two point set $\{0,1/n\}$. The fact that $\hat u$ is continuous on Ω^* implies Φ_z has at least two components, one of which contains χ and one of which contains η, a contradiction.

Conversely, let $Q(U,\chi)=Q(U,\eta)$ for each neighborhood U of z, and let $\{z_\alpha\}$ and $\{y_\beta\}$ be nets in Ω such that $\hat z_\alpha\to\chi$ and $\hat y_\beta\to\eta$. Suppose that χ and η are in different components Ψ and Π of Φ_z. Since Φ_z is a closed subset of Δ, the component Ψ and the set $\Phi_z\setminus\Psi$ are also closed. By the Urysohn Lemma, there is a $g\in C(\Omega^*)$ such that $g|\Psi\equiv 1$ and $g|(\Phi_z\setminus\Psi)\equiv 0$. Because $\hat A$ is dense in $C(\Omega^*)$, there is $\hat v\in\hat A$ so that the range of $\hat v$ restricted to Ψ lies in $(\frac{2}{3},\frac{4}{3})$ and the range of $\hat v$ restricted to $\Phi_z\setminus\Psi$ lies in $(-\frac{1}{3},\frac{1}{3})$. Because χ is equivalent to η, $Q\big(\mathbf{B}(z,1/n),\chi\big)=Q\big(\mathbf{B}(z,1/n),\eta\big)=Q_n$ for each positive integer n. Thus $\{z_\alpha\}$ is eventually in Q_n and $v(z_\alpha)$ is eventually greater than $\frac{1}{2}$, while $\{y_\beta\}$ is eventually in Q_n and $v(y_\beta)$ is eventually less than $\frac{1}{2}$. Because Q_n is connected and v is continuous, for each n there exists $x_n\in Q_n$ such that $v(x_n)=\frac{1}{2}$. Corollary 4.7 then implies that $\frac{1}{2}$ lies in the range of $\hat v$ on Φ_z, a contradiction. □

Corollary 7.3. *If Ω is locally connected at $z\in\partial\Omega$, then Φ_z is connected.*

Proof. By Theorem 7.2, it is enough to show that if $\chi,\eta\in\Phi_z$ then $Q(U,\chi)=Q(U,\eta)$ for each neighborhood U of z. Let U be a neighborhood of z. Then because Ω is locally connected on the boundary there exists a neighborhood V of z such that $V\subset U$ and $V\cap\Omega$ is connected. Thus $Q(V,\chi)=Q(V,\eta)$. By definition, both $Q(U,\chi)$ and $Q(U,\eta)$ are components of $U\cap\Omega$ that contain $Q(V,\chi)$; therefore they are the same. □

Theorem 7.4. *Let Ω be finitely connected on the boundary. The set C of components of boundary fibers in Δ is identical to the set $\mathscr{P}$ of prime ends of Ω.*

Proof. We define the function $\kappa\colon \mathscr{P} \to C$ by the rule $\kappa(\tilde{\gamma}) = \Psi$ if and only if $h(\gamma) = \pi(\Psi)$ and, for each $\chi \in \Psi$ and neighborhood U of z in $\overline{\mathbf{R}}^n$, $P(U,\gamma) = Q(U,\chi)$. The fact that κ is well-defined and injective follows from Theorem 7.2. We show that κ is surjective and hence a bijection. Let $\Psi \in C$ and $\chi \in \Psi$; then $\{Q(\mathbf{B}(z,1/j),\chi)\} = \{Q_j\}$ is a sequence of connected open sets in $\mathbf{R}^n$ with $Q_{j+1} \subset Q_j$ for each $j \in \mathbf{N}$. For each j, let x_j be a point in Q_j. Since Q_j is an open connected set in $\mathbf{R}^n$, there is a path γ_j joining x_j to x_{j+1} in Q_j. Let γ be the path defined by joining γ_1 to γ_2 to γ_3, etc. Then $h(\gamma) = z$. If U is a neighborhood of z, then there is m large enough so that $Q_m \subset U$. By definition, each Royden net $\{z_\alpha\}$ corresponding to χ is eventually in the connected set $Q_m \subset U$, so $Q(U,\chi)$ must be the component of $U \cap \Omega$ containing Q_m. On the other hand, Q_m contains a subendcut of γ, and therefore $P(U,\gamma)$ is the component of $U \cap \Omega$ containing Q_m. Thus $P(U,\gamma) = Q(U,\chi)$. □

8. The Royden boundary and Stone–Cech compactification of N

Finally, we examine the topology of individual boundary fibers. Because each sequence tending to $\partial\Omega$ is an interpolating sequence for $A(\Omega)$, Corollary 4.9 highlights how differently nets converging to a boundary point may behave with respect to $A(\Omega)$ than sequences converging to a boundary point. Boundary fibers are determined by nets, so one might guess that whenever $z \in \partial\Omega$, the fiber Φ_z is a large and complicated set. This is true. If $\beta\mathbf{N}$ denotes the Stone–Cech compactification, then each such fiber contains a homeomorphic copy of $\beta\mathbf{N} \setminus \mathbf{N}$.

Up to homeomorphism, $\beta\mathbf{N}$ is the unique compactification of $\mathbf{N}$ on which each real-valued bounded continuous function on $\mathbf{N}$ can be uniquely extended. The space $\beta\mathbf{N}$ can also be characterized as the compactification of $\mathbf{N}$ that has the properties outlined in the following theorem.

Theorem 8.1 [Ga, Theorem 1.4, p. 186]. *Let Y be a compact Hausdorff space and let $\tau\colon \mathbf{N} \to Y$ be a continuous mapping. Then the mapping τ has a unique continuous extension $\overline{\tau}\colon \beta\mathbf{N} \to Y$. If $\tau(\mathbf{N})$ is dense in Y and if the images of disjoint subsets of $\mathbf{N}$ have disjoint closures in Y, then the extension $\overline{\tau}$ is a homeomorphism of $\beta\mathbf{N}$ onto Y.*

The Stone–Cech compactification of $\mathbf{N}$ is a huge space. For example, if Z is any infinite closed subset of $\beta\mathbf{N}$, then $\mathrm{card}(\mathrm{Z}) = \mathrm{card}(\mathrm{I}^{\mathrm{I}})$ where $I = [0,1]$, [D, p. 244]. In fact, $\beta\mathbf{N}$ can be mapped onto *any* separable compact Hausdorff space; however, no point of $\beta\mathbf{N} \setminus \mathbf{N}$ can be exhibited concretely [Ga, p. 187]. This is unfortunate for the study of fibers in the Royden boundary because of the following theorem.

Theorem 8.2. *For each $z \in \partial\Omega$, Φ_z contains a homeomorphic image of $\beta\mathbf{N} \setminus \mathbf{N}$.*

Proof. Let $\{z_j\}$ be a sequence of distinct points in Ω such that $z_j \to z \in \partial\Omega$, and let Y be the closure of $\{\hat{z}_j\}$ in Ω^*. Then as a subspace of Ω^*, Y is compact and Hausdorff. We define $\tau\colon \mathbf{N} \to Y$ by $\tau(j) = \hat{z}_j$ for each $j \in \mathbf{N}$, and thus $\tau(\mathbf{N}) = \{\hat{z}_j\}$ is dense in Y.

If $S = \{s_1, s_2, \ldots\}$ and $Q = \{q_1, q_2, \ldots\}$ are disjoint subsets of $\mathbf{N}$, then because $\{z_j\}$ is an interpolating sequence, there exists $u \in A(\Omega)$ for which $u(z_{s_i}) = 1$ and $u(z_{q_k}) = 0$ for each i and k. This means $\hat{u}(\hat{z}) = 1$ for each $\hat{z} \in \tau(S)$; and since $\hat{u} \in C(\Omega^*)$, $\chi \in \overline{\tau(S)}$ implies $\chi(u) = \hat{u}(\chi) = 1$. Similarly, $\eta \in \overline{\tau(Q)}$ implies $\eta(u) = 0$. Thus $\overline{\tau(S)} \cap \overline{\tau(Q)} = \emptyset$, and by Theorem 8.1, τ has a homeomorphic extension $\overline{\tau}$ of $\beta\mathbf{N}$ onto Y. Because $z_j \to z$ and Y is the closure of $\{\hat{z}_j\}$ minus any finite set, it follows that $\hat{\sigma}_z$ is identically 0 on $Y \setminus \{\hat{z}_j\}$; i.e. $\chi(\sigma_z) = 0$ for each $\chi \in Y \setminus \{\hat{z}_j\}$, so

$$\overline{\tau}(\beta\mathbf{N} \setminus \mathbf{N}) = Y \setminus \{\hat{z}_j\} \subset \Phi_z.$$

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