

**ON A NEW SYSTEM OF TWO GENERAL  
EQUATIONS OF CURVATURE**

**By**

**William Rowan Hamilton**

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Edited by David R. Wilkins

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## NOTE ON THE TEXT

This edition is based on the original text published posthumously in volume 9 of the *Proceedings of the Royal Irish Academy*.

The following obvious typographical errors have been corrected:—

before equation (a), a full stop (period) has been changed to a colon;

in equation (k), ‘ $(Z - x)$ ’ has been corrected to ‘ $(Z - z)$ ’;

in equation (o), ‘ $C = eE'' - e'E$ ’ has been corrected to ‘ $C = eE' - e'E$ ’;

equation (q) in the original text was given as

$$(eR^{-1} - eK^{-1})(e''R^{-1} - e''K^{-1}) = (e'R^{-1} - eK^{-1})^2;$$

in equation (r), ‘ $= 0$ ’ has been appended to the polynomial ‘ $R^{-2} - FR^{-1} + G$ ’;

in equation (w), the equality sign  $=$  has been added.

David R. Wilkins  
Dublin, March 2000

ON A NEW SYSTEM OF TWO GENERAL EQUATIONS OF CURVATURE,

Including as easy consequences a new form of the Joint Differential Equation of the Two Lines of Curvature, with a new Proof of their General Rectangularity; and also a new Quadratic for the Joint Determination of the Two Radii of Curvature: all deduced by Gauss's Second Method, for discussing generally the Properties of a Surface; and the latter being verified by a Comparison of Expressions, for what is called by him the Measure of Curvature.

Sir William Rowan Hamilton

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1. Notwithstanding the great beauty and importance of the investigations of the illustrious GAUSS, contained in his *Disquisitiones Generales circa Superficies Curvas*, a Memoir which was communicated to the *Royal Society of Göttingen* in October, 1827, and was printed in Tom. vi. of the *Commentationes Recentiores*, but of which a Latin reprint has been since very judiciously given, near the beginning of the Second Part (Deuxième Partie, Paris, 1850) of LIOUVILLE'S *Edition*\* of MONGE, it still appears that there is room for some not useless Additions to the Theory of *Lines and Radii of Curvature*, for any given Curved Surface, when treated by what Gauss calls the *Second Method* of discussing the *General Properties of Surfaces*. In fact, the *Method* here alluded to, and which consists chiefly in treating the three co-ordinates of the *surface* as being so many *functions* of two independent variables, does not seem to have been used *at all* by Gauss, for the determination of the *Directions of the Lines of Curvature*; and as regards the *Radii of Curvature* of the *Normal Sections* which touch these *Lines* of Curvature, he appears to have employed the *Method*, *only for the Product*, and *not also* for the *Sum*, of the *Reciprocals*, of those *Two Radii*.

2. As regards the *notations*, let  $x, y, z$  be the rectangular co-ordinates of a point P upon a surface ( $S$ ), considered as *three* functions of *two* independent variables,  $t$  and  $u$ ; and let the 15 partial derivatives, or 15 partial differential coefficients, of  $x, y, z$  taken with respect to  $t$  and  $u$ , be given by the nine differential expressions:

$$(a) \dots \begin{cases} dx = x' dt + x'' du; & dx' = x'' dt + x''' du; & dx'' = x''' dt + x'''' du; \\ dy = y' dt + y'' du; & dy' = y'' dt + y''' du; & dy'' = y''' dt + y'''' du; \\ dz = z' dt + z'' du; & dz' = z'' dt + z''' du; & dz'' = z''' dt + z'''' du. \end{cases}$$

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\* The foregoing dates, or references, are taken from a note to page 505 of that Edition.

3. Writing also, for abridgment,

$$(b) \dots e = x'^2 + y'^2 + z'^2; \quad e' = x'x_1 + y'y_1 + z'z_1; \quad e'' = x_1^2 + y_1^2 + z_1^2$$

we shall have

$$(c) \dots ee'' - e'^2 = K^2,$$

if

$$(d) \dots K^2 = L^2 + M^2 + N^2,$$

and

$$(e) \dots L = y'z_1 - z'y_1; \quad M = z'x_1 - x'z_1; \quad N = x'y_1 - y'x_1;$$

so that

$$(f) \dots Lx' + My' + Nz' = 0, \quad Lx_1 + My_1 + Nz_1 = 0.$$

Hence  $K^{-1}L$ ,  $K^{-1}M$ ,  $K^{-1}N$  are the *direction-cosines* of the *normal* to the surface ( $S$ ) at P; and if  $x$ ,  $y$ ,  $z$  be the co-ordinates of any *other* point Q of the same normal, we shall have the equations

$$(g) \dots K(X - x) = LR; \quad K(Y - y) = MR; \quad K(Z - z) = NR;$$

with

$$(h) \dots R^2 = (X - x)^2 + (Y - y)^2 + (Z - z)^2;$$

where  $R$  denotes the normal line PQ, considered as changing sign in passing through zero.

4. The following, however, is for some purposes a more convenient *form* (comp. (f)) of the *Equations of the Normal*;

$$(i) \dots (X - x)x' + (Y - y)y' + (Z - z)z' = 0;$$

$$(j) \dots (X - x)x_1 + (Y - y)y_1 + (Z - z)z_1 = 0.$$

Differentiating these, as if  $X$ ,  $Y$ ,  $Z$  were constant, that is, treating the point Q as an intersection of two consecutive normals, we obtain these two other equations,

$$(k) \dots \begin{cases} (X - x) dx' + (Y - y) dy' + (Z - z) dz' = x' dx + y' dy + z' dz; \\ (X - x) dx_1 + (Y - y) dy_1 + (Z - z) dz_1 = x_1 dx + y_1 dy + z_1 dz. \end{cases}$$

If, then, we write, for abridgment,

$$(l) \dots \begin{cases} v = du : dt; & E = Lx'' + My'' + Nz''; \\ E' = Lx'_1 + My'_1 + Nz'_1; & E'' = Lx_{11} + My_{11} + Nz_{11}; \end{cases}$$

we shall have, by (a) (b) (g), the two important formulæ:

$$(m) \dots R(E + E'v) = K(e + e'v); \quad R(E' + E''v) = K(e' + e''v);$$

which we propose to call the two general *Equations of Curvature*.

5. In fact, by elimination of  $R$ , these equations (m) conduct to a *quadratic in  $v$* , of which the roots may be denoted by  $v_1$  and  $v_2$ , which first presents itself under the form,

$$(n) \dots (e + e'v)(E' + E''v) = (e' + e''v)(E + E'v),$$

but may easily be thus transformed,

$$(o) \dots \begin{cases} Av^2 - Bv + C = 0, \text{ or } A du^2 - B dt du + C dt^2 = 0, \\ \text{with } A = e'E'' - e''E', \quad B = e''E - eE'', \quad C = eE' - e'E; \end{cases}$$

so that we have the following *general relation*,

$$(p) \dots eA + e'B + e''C = 0,$$

(of which we shall shortly see the geometrical signification), between the *coefficients*,  $A, B, C$ , of the *joint differential equation* of the system of the two *Lines of Curvature* on the surface.

6. The root  $v_1$  of the quadratic (o) determines the *direction* of what may be called the *First Line of Curvature*, through the point P of that surface; and the *First Radius of Curvature*, for the same point P, or the radius  $R_1$  of curvature of the *normal section* of the surface which *touches* that *first line*, may be obtained from *either* of the two equations (m), as the value of  $R$  which corresponds in that equation to the value  $v_1$  of  $v$ . And in like manner, the *Second Radius of Curvature* of the same surface at the same point has the value  $R_2$ , which answers to the value  $v_2$  of  $v$ , in each of the same two *Equations of Curvature* (m). We see, then, that this *name* for those two equations is justified by observing that when the two independent variables  $t$  and  $u$  are given or known; and therefore also the seven functions of them, above denoted by  $e, e', e'', E, E', E''$ , and  $K$ . The equations (m) are satisfied by *two* (but *only two*) *systems of values*,  $v_1, R_1$ , and  $v_2, R_2$ , of (I.) the *differential quotient*  $v$ , or  $\frac{du}{dt}$ , which determines the *direction* of a *line of curvature* on the surface; and (II.) the symbol  $R$ , which determines (comp. No. 4) at once the *length* and the *direction*, of the *radius of curvature*, corresponding to that *line*.

7. Instead of eliminating  $R$  between the two equations (m), we may *begin* by eliminating  $v$ ; a process which gives the following quadratic in  $R^{-1}$  (the curvature):—

$$(q) \dots (eR^{-1} - EK^{-1})(e''R^{-1} - E''K^{-1}) = (e'R^{-1} - E'K^{-1})^2;$$

or (r) ..  $R^{-2} - FR^{-1} + G = 0$ ; where (because  $ee'' - e'^2 = K^2$ ),

$$(s) \dots F = R_1^{-1} + R_2^{-1} = (eE'' - 2e'E' + e''E)K^{-3}, \text{ and}$$

$$(t) \dots G = R_1^{-1}R_2^{-1} = (EE'' - E'^2)K^{-4}.$$

We ought, therefore, as a *First General Verification*, to find that this last expression, which may be thus written,

$$(u) \dots G = R_1^{-1} R_2^{-1} = \frac{EE'' - E'E'}{(L^2 + M^2 + N^2)^2},$$

agrees with that reprinted in page 521 of Liouville's Monge, for what Gauss calls the *Measure of Curvature* ( $k$ ) of a *Surface*; namely,

$$(v) \dots k = \frac{DD'' - D'D'}{(AA + BB + CC)^2};$$

which accordingly it evidently does, because our symbols  $L M N A B C$  represent the combinations which he denotes by  $A B C D D' D''$ .

8. As a *Second General Verification*, we may observe that if  $I$  be the *inclination* of any *linear element*,  $du = v dt$ , to the *element*  $du = 0$ , at the point  $P$ , then

$$(w) \dots \tan I = \frac{Kv}{e + e'v};$$

and therefore, that if  $H$  be the *angle* at which the *second crosses the first*, of any *two lines* represented *jointly* by such an equation as

$$(x) \dots Av^2 - Bv + C = 0, \text{ with } v_1 \text{ and } v_2 \text{ for roots, then}$$

$$(y) \dots \tan H = \tan(I_2 - I_1) = \frac{K(B^2 - 4AC)^{\frac{1}{2}}}{eA + e'B + e''C};$$

so that the *Condition of Rectangularity* ( $\cos H = 0$ ), for any *two* such lines, may be thus written:

$$(z) \dots eA + e'B + e''C = 0.$$

But this *condition* (z) had already occurred in No. 5, as an equation (p) which is satisfied generally by the *Lines of Curvature*; we see therefore anew, by this analysis, that those *lines* on any *surface* are in general (as is indeed well known) *orthogonal* to each other.

9. Finally, as a *Third General Verification*, we may assume  $x$  and  $y$  themselves (instead of  $t$  and  $u$ ), as the two independent variables of the problem, and then, if we use *Monge's Notation* of  $p, q, r, s, t$ , we shall easily recover all his leading results respecting *Curvatures of Surfaces*, but by transformations on which we cannot here delay.