# ON THE EXISTENCE OF A SYMBOLIC AND BIQUADRATIC EQUATION WHICH IS SATISFIED BY THE SYMBOL OF LINEAR OR DISTRIBUTIVE OPERATION ON A QUATERNION

**By**

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*On the Existence of a Symbolic and Biquadratic Equation which is satisfied by the Symbol of Linear or Distributive Operation on a Quaternion. By* Sir William Rowan Hamilton, *LL.D. &c.**

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1. As early as the year 1846, I was led to perceive the existence of a certain *symbolic* and *cubic equation*, of the form

$$0 = m - m'\phi + m''\phi^2 - \phi^3, \tag{1}$$

in which $\phi$ is used as a symbol of *linear* and *vector operation* on a *vector*, so that $\phi\rho$ denotes a vector depending on $\rho$, such that

$$\phi(\rho + \rho') = \phi\rho + \phi\rho', \tag{2}$$

if $\rho$ and $\rho'$ be any two vectors; while $m$, $m'$ and $m''$ are *three scalar constants*, depending on the *particular* form of the linear and vector function $\phi\rho$, or on the (scalar or vector) constants which enter into the composition of that function. And I saw, of course, that the problem of *inversion* of such a *function* was at once given by the formula

$$m\phi^{-1} = m' - m''\phi + \phi^2, \tag{3}$$

—the required assignment of the inverse function, $\phi^{-1}\rho$, being thus reduced to the performance of a limited number of *direct operations*.

2. Quite recently I have discovered that the far more general *linear* (or distributive) and *quaternion function of a quaternion* can be *inverted*, by an analogous process, or that there always exists, for any *such* function $fq$, satisfying the condition

$$f(q + q') = fq + fq', \tag{4}$$

where $q$ and $q'$ are any two quaternions, a *symbolic* and *biquadratic* equation, of the form

$$0 = n - n'f + n''f^2 - n'''f^3 + f^4, \tag{5}$$

in which $n$, $n'$, $n''$, and $n'''$ are *four scalar constants*, depending on the particular composition of the linear function $fq$; and that therefore we may write generally this *Formula of Quaternion Inversion*,

$$nf^{-1} = n' - n''f + n''f^2 - f^3. \tag{6}$$

* Communicated by the Author.

3. As it was in the Number of the Philosophical Magazine for July 1844 that the first *printed* publication of the Quaternions occurred (though a paper on them had been previously read to the Royal Irish Academy in November 1843), I have thought that the Editors of the Magazine might perhaps allow me thus to put on record what seems to myself an important addition to the theory, and that they may even allow me to add, in a Postscript to this communication, so much as may convey a distinct conception of the *Method* which I have pursued.

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