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Erdős, Pál; Hajnal, András; Rado, R.
Partition relations for cardinal numbers (In English)
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Small Greek letters denote ordinal numbers, small Roman letters denote cardinal numbers (i.e. initial ordinal), always $p, r, s < \omega_0$, $|X|$ is the cardinality of X, $[X]^r$ denotes the set of all r element subsets of X. The partition relations I, II, III are defined as follows. The relation I: $a \to (b_\nu)^r_{\nu<\lambda}$ holds true if and only if for every partition $[X]^r = \bigcup_{\nu<\lambda} J_\nu$, $|X| = a$, there is a $\nu_0 < \lambda$ and a subset $Y \subseteq X$ such that $|Y| = b$ and $[Y]^r \subseteq J_{\nu_0}$. The relation II: $a \to (b_\nu)^{<\aleph_0}_{\nu<\lambda}$ means that for every partition $[X]^{<\aleph_0} = \bigcup_{\nu<\lambda} J_\nu$, where $[X]^{<\aleph_0} = \bigcup_{r<\omega_0} [X]^r$, there exista a $\nu_0 < \lambda$ and a subset $Y \subseteq X$, $|Y| = b_{\nu_0}$ and $[Y]^{<\aleph_0} \subseteq J_{\nu_0}$. The relation III:

$$\begin{pmatrix} a_0 \\ \vdots \\ a_s \end{pmatrix} \to \begin{pmatrix} b_{0\nu} \\ \vdots \\ b_{s\nu} \end{pmatrix}^{r_0,\dots,r_s}_{\nu<\lambda}$$

is equivalent to the following condition. Let $|X_p| = a_p$ for $p \le s$, X_p are disjoint, $[X_0, ..., X_s]^{r_0,\dots r_s} = \{X : X \subseteq X_0 \cup \cdots \cup X_s,\ |X \cap X_p| = r_p$ for $p \le s\} = \bigcup_{\nu<\lambda} J_\nu$. Then there exist sets $Y_r \subseteq X_r$, for $r \le s$ and a $\nu_0 < \lambda$ such that $|Y_r| = b_{r\nu_0}$ for $r \le s$ and

$$[Y_0, ..., Y_s]^{r_0,\dots,r_s} \subseteq J_{\nu_0}.$$

"In this paper our first major aim is to discuss as completely as possible the relation I. Our most general results in this direction are stated in Theorems I and II, ... If we disregard cases when among the given cardinals there occur inaccessible numbers greater than $\aleph_0$, and if we assume the General Continuum Hypothesis, then our results are complete for $r = 2$, ... It seems that there are only two essentially different methods for obtaining positive partition formulae: those given in Lemma 1 and those given in Lemma 3 ... In Lemma 5 we state powerful new methods for constructing examples of particular partitions which yield negative I-relation. ... Our second major aim is an investigation of the polarized partition relation III."

The exact formulation of the Lemma 1 is complicated; its contents may be shortly formulated as follows: in every sufficiently great tree, in which from every edge goes out a small number of branches, there is a large branche.

The simplest canonization lemma (the Lemma 3 proved using the Generalized Continuum Hypothesis) may be stated as follows: Let $|S| = a > a'$ (a' is the smallest cardinal with which a is cofinal); $r \ge 1$, $a = \sup\{a_\xi < a'\}$, $a_{\xi_1} < a_{\xi_2}$ for $\xi_1 < \xi_2 < a'$, $[S]^r = \bigcup_{\nu<\lambda} J_\nu$, $\lambda < a$. Then there are disjoint sets S_σ, $\sigma < a'$, $|S_\sigma| = a_\sigma$, $S_\sigma \subseteq S$ and for $X, Y \in \left[\bigcup_{\sigma<a'} S_\sigma\right]^r$, the relations $|X \cap S_\sigma| = |Y \cap S_\sigma|$ for $\sigma < a'$ are equivalent to the condition: there is a $\nu_0 < \lambda$ such that $X, Y \in J_{\nu_0}$.

Define $\alpha \dot{-} 1 = \alpha$ for α limit $\alpha \dot{-} 1 = \beta$ if and only if $\alpha = \beta + 1$, $\mathrm{cr}(\alpha) = \mathrm{cf}(\mathrm{cf}(\alpha \dot{-} 1)$. Let us denote:

(R) $\aleph_{\beta+(r-2)} \to (b_\xi)^r_\xi < \lambda$,
(IA) $b_0 = \aleph_\beta$,
(IB) $b_\xi < \aleph_\beta$ for $\xi < \lambda$,
(CA) $\prod_{1\le\xi<\lambda} b_\xi \le \aleph_{cr(\beta)}$,
(CB) $\prod_{\xi<\lambda} b_\xi < \aleph_\beta$,
(D) $r \ge 3$, $\beta > \mathrm{cf}(\beta) > \mathrm{cf}(\beta)\dot{-}1 > \mathrm{cr}\beta$, $b_\xi < \aleph_0$ for $1 \le \xi < \lambda$.
The first main theorem may be stated as follows. Let $\lambda \ge 2$, $2 \le r < b_\xi \le \aleph_\beta$ for $\xi < \lambda$. Assuming the Generalized Continuum Hypothesis we have:
(i) If (IA) holds, (D) does not holds, then (R) implies (CA).
(ii) If (IA) holds and $b_1 \ge \aleph_0$, then (R) implies (CA).
(iii) If (IA) holds and $\aleph'_\beta$ is not inaccessible, then (CA) implies (R).
(iv) If (IA) holds and $b_\xi < \aleph'_\beta$ for $0 < \xi < \lambda$ then (CA) implies (R).
(v) If (IB) holds, then (CB) is equivalent to (R).
Let us denote:
(IIA) $b_0 > \aleph_{\alpha\dot{-}(r-2)}$.
(IIB) $b_\xi \le \aleph_\gamma$, $\xi < \lambda$, $\alpha = \gamma + s$, γ limit and $s < r-2$.
(IIC1) $b_0 = \aleph_{\alpha\dot{-}(r-2)}$,
(IIC2) $b_\xi < \aleph_{\alpha\dot{-}(r-2)}$ for $\xi < \lambda$.
(R0) $\aleph_\alpha \to (b_\xi)^r_{\xi<\lambda}$.
The second main theorem: Let $\lambda \ge 2$, $2 \le r < b_\xi \le \aleph_\alpha$ for $\xi < \lambda$.
Assuming the Generalized Continuum Hypothesis we have:
(i) If (IIA) holds, then (R0) is false.
(ii) If (IIB) and (IIC1) hold, (R0) implies that $\aleph_{\alpha\dot{-}(r-2)}$ is inaccessible.
(iii) If (IIB) and (IIC2) hold, then (R0) is equivalent to the condition $\prod_{\xi<\lambda} b_\xi < \aleph_{\alpha\dot{-}(r-2)}$.
The proofs are based on Lemmas 1, 2, 3 and 5. The Lemma 2 and 5 are the stepping-up and stepping-down Lemmas respectively, i.e. they are of the form "if $a \to (b_\xi)^r_{\xi<\lambda}$, then $a^+ \to (b_\xi+1)^{r+1}_{\xi<\lambda}$" and "if $a \not\to (b_\xi)^r_{\xi<\lambda}$, then $2^a \not\to (b_\xi+1)^{r+1}_{\xi<\lambda}$", respectively (of course, under some assumptions).
A great part of the paper is devoted to the study of relations IV and V. The relation IV: $a \to [b_\xi]^r_{\xi<c}$ (relation V: $a \to [b]^r_{c,d}$) is equivalent to the condition: whenever $|S| = a$, $[S]^r = \bigcup_{\xi<c} J_\xi$, where the J_ξ are disjoint, then there are a set $X \subseteq S$ and a number $\xi_0 < c$ (a set $D \subseteq c$) such that $|X| = b_{\xi_0}$ and $[X]^r \cap J_{\xi_0} = \emptyset$ ($|X| = b$, $|D| \le a$ and $[X]^r \subseteq \bigcup_{\xi\in D} J_\xi$). Some results (assuming the Generalized Continuum Hypothesis):
(i) $\aleph_{\alpha+1} \not\to [\aleph_{\alpha+1}]^2_{\aleph_{\alpha+1}}$ for any α.
(ii) Let $r \ge 2$ and $\alpha > \mathrm{cf}(\alpha)$. Then $\aleph_\alpha \not\to [\aleph_\alpha]^r_{2^{r-1}}$.
(iii) If $\aleph'_\alpha$ is $\aleph_0$ or a measurable cardinal, then $\aleph_\alpha \to [\aleph_\alpha]^r_c$ for $c > 2^{r-1}$ and $\aleph_\alpha \to [\aleph_\alpha]^r_{c2^{r-1}}$ for $c < \aleph_\alpha$.
(iv) $\aleph_2 \to [\aleph_0, \aleph_1, \aleph_1]^3$.
On the other hand, there are many open problems, e.g. $\aleph_2 \to [\aleph_1]^3_4$?, $\aleph_3 \to [\aleph_1]^2_{\aleph_2,\aleph_0}$?
In the second part, the authors investigate the polarized partition relation $\binom{a}{b} \to \begin{pmatrix} a_0, & a_1 \\ b_0, & b1 \end{pmatrix}$, i. e. a special case of the relation III. A complete discussion is given, however, the results are not complete. Many other relations and problems are studied, but it is impossible to give a full list of them here.

The paper is rather difficult to read and gives the impression of a condensed version of a monography.

L.Bukovský

Classification:
05D10 Ramsey theory
03E05 Combinatorial set theory (logic)
04A20 Combinatorial set theory
03-02 Research monographs (mathematical logic)
05E10 Tableaux, etc.
04A10 Ordinal and cardinal numbers; generalizations